PROBLEM SET II (DUE NOV. 9, 2018)

1. Problem 0

Show that intersection of countable number of k-cells, say $\{I_n\}_{n\geq 1}$ is nonempty if $I_{n+1} \subset I_n$ for any $n \geq 1$.

2. Problem 1

Show that there can be only finite or countable number of pairwise disjoint symbol "8"'s drawn on the plane.

3. Problem 2

Let $\{0,1\}^{\mathbb{N}}$ be the Cantor group, i.e., all possible sequences of the type $(0,0,1,1,0,\ldots)$ consisting of 0's and 1's. Let $\mathcal{P}(\mathbb{N})$ denote the collection of all subsets of natural numbers $\mathbb{N} = \{1,2,\ldots\}$. Construct $1-1$, onto map $f : \{0,1\}^{\mathbb{N}} \mapsto \mathcal{P}(\mathbb{N})$.

4. Problem 3

I suggest to solve **all** problems listed in the exercise section of Chapter 2. Here are some of them: 2, 5, 8, 9, 10 (very strange problem), 11, 13, 14, 17 (looks like an interesting problem), 21(c), 22, 23, 24, 25, (these 23, 24, 25, 26 are together, in some literature people define compact sets by saying that every infinite subset has a limit point in it) 26, 29, 30 Chapter 2.

On our midterm exam there will be 5 problems: 4 of them will be from Rudin.

5. Bonus problem 1

Given an integer $n \geq 1$, suppose we want to find n different vectors $v_1, v_2, \ldots, v_n$ in $\mathbb{R}^n$ with coordinates $+1$ or -1 such that the inner product of any two of them is zero. Can one solve this problem when $n = 6$? When $n = 8$? Notice that when $n = 2$ one can take $v_1 = (1,1)$ and $v_2 = (-1,1)$. Then clearly $v_1 \cdot v_2 = 0$.

6. Bonus problem 2

Let $n \geq 1$ be a fixed integer. Suppose we are given on the plane n red vertices and n blue vertices. Suppose there are edges joining red and blue vertex, and we do not know how many edges are there but there is the following constraint: for **any** $k \leq n$, and **any** k-subset of red vertices (call it A) there exist at least k blue vertices (call it B) such that for any vertex $b \in B$ there is an edge joining b to a vertex $a \in A$. Does this constraint imply that we can split red and blue vertices in n pairs of vertices (red,blue) such that vertices from each pair are joined by an edge (each vertex can be only in one pair).

7. Why cuts (real numbers) are uncountable?

Let us construct 1-1 map f from the set R_+ of all nonnegative cuts $x \geq 0$ to the set of all objects of the form

$$n_0.n_1n_2n_3\ldots,$$

where $n_0 \geq 0$ is any integer and $n_1.n_2,\ldots$ can be nonnegative integers taking values $0,1,2,\ldots,9$.

Definition. *All such sequences we denote by* $\mathbb{W}$.

Also notice that we can consider negative cuts as well just by putting negative sign everywhere.
Here starts the construction.

Take any $x \in R_+$, $x \geq 0$. Let $n_0 \geq 0$ be the largest integer such that $n_0 \leq x$. Now let $n_1 \geq 0$ be the largest integer such that $n_0 + \frac{n_1}{10} \leq x$. Notice that $0 \leq n_1 \leq 9$ (why n_1 cannot be greater than 9?). And we iterate this process: suppose $n_0, n_1, \dots, n_k$ are already constructed then we define $n_{k+1} \geq 0$ to be the largest integer such that

$$n_0 + \frac{n_1}{10} + \dots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \leq x.$$

Thus we have a map $x \mapsto n_0.n_1n_2\dots$, i.e., $f : R_+ \mapsto \mathbb{W}$.

Why is f one-to-one?

For this it would be enough to show that $x = \sup E$ where

$$E = \left\{ n_0 + \frac{n_1}{10} + \dots + \frac{n_k}{10^k} \right\}_{k \geq 0}.$$

Indeed, this would mean that if there are two reals $x, y \geq 0$, $x \neq y$ such that their image gives one and the same object $n_0.n_1n_2\dots$, then $x = \sup E$, and $y = \sup E$ which would be a contradiction (since $\sup E$ is unique). Thus it suffices to verify the property that $x = \sup E$. Notice that E is nonempty, and E is upper bounded (for any $e \in E$ we have $e \leq x$). Therefore $\sup E$ exists in R. Denote $t \stackrel{\text{def}}{=} \sup E$. Next we show that $t = x$. It is clear that $x \geq t$ since x is an upper bound for E and t is the least upper bound. Next, assume the contrary that $t < x$. Set $\varepsilon = x - t > 0$. Let $\ell \geq 0$ be such that $\frac{1}{10^\ell} < \frac{\varepsilon}{2}$. Clearly

$$n_0 + \frac{n_1}{10} + \dots + \frac{n_\ell}{10^\ell} \leq t$$

Therefore

$$n_0 + \frac{n_1}{10} + \dots + \frac{n_\ell + 1}{10^\ell} \leq t + \frac{1}{10^\ell} < t + \varepsilon = x$$

But this means that n_ℓ was not the largest nonnegative integer such that

$$n_0 + \frac{n_1}{10} + \dots + \frac{n_\ell}{10^\ell} \leq x.$$

A contradiction. Thus $x = \sup E$.

Is f onto? The answer is NO.

Indeed, consider the following sequence $0.999\dots$. And show that one cannot find a cut $x \in R_+$ such that $f(x) = 0.999\dots$. The "right" candidate should be $x = 1$ but for this one we have $f(1) = 1.000\dots$. Thus we see that there are "bad" elements in $\mathbb{W}$. Let us call an object $n_0.n_1n_2\dots \in \mathbb{W}$ bad if starting at some point $k \geq 1$ we have $9 = n_k = n_{k+1} = n_{k+2} = \dots$. In other words these are those sequences who have a "tail" consisting only by 9's. Let us denote the set of all bad elements of $\mathbb{W}$ by B.

Problem 7.1. *Show that $f : R_+ \mapsto \mathbb{W} \setminus B$ is 1-1, and onto.*

Let us split the solution of this problem into several steps. To show that f is 1-1 it will be enough to prove the following lemma

Lemma 7.1. *For each $b \in B$ no $x \in R$ satisfies $f(x) = b$.*

Proof. Here is the sketch: suppose for some $b \in B$ there is x such that $f(x) = b$. We know that starting at some point b has the tail of $9's$. If $b = n_0.999\dots$ show that the supremum of the set of numbers

$$E = \left\{ n_0, \quad n_0 + \frac{9}{10}, \quad n_0 + \frac{9}{10} + \frac{9}{10^2}, \dots \right\}$$

is $n_0 + 1 = x$. Indeed, clearly $n_0 + 1$ is un upper bound. It remains to show that it is the least upper bound (Exercise). On the other hand $f(x) = f(n_0 + 1) = (n_0 + 1).000\dots \neq n_0.999\dots$.

Now, if

$$b = n_0.n_1n_2\dots n_k 9999\dots$$

where $n_k < 9$ and $k \geq 1$, then show that the supremum of the corresponding set of E is $b = n_0.n_1n_2\dots(n_k+1)000\dots$. Again this is left as an exercise. □

The lemma together with the previous reasoning implies that the map $f : R_+ \mapsto \mathbb{W} \setminus B$ is one-to-one

Finally it remains to solve the following problem.

Problem 7.2. *Show that $f : R_+ \mapsto \mathbb{W} \setminus B$ is onto.*

Here we need to show that given any object $m_0.m_1m_2\dots$ (here $m_0 \geq 0$ is any integer and $m_1, m_2, \dots$ take values $0, 1, 2\dots, 9$) we can find a nonnegative real x such that $f(x) = m_0.m_1m_2$. Indeed, let U to be the set of all numbers $m_0 + \frac{m_1}{10} + \dots + \frac{m_k}{10^k}$ for all $k \geq 0$. Then U is nonempty, and it is bounded from above. Indeed, let us show that for any $e \in U$ we have $e \leq m_0 + 1$. We have

$$e = m_0 + \frac{m_1}{10} + \dots + \frac{m_\ell}{10^\ell}$$

for some $\ell \geq 0$. If $\ell = 0$ there is nothing to prove. Assume $\ell \geq 1$. Then

$$e \leq m_0 + \frac{9}{10}\left(1 + \dots + \frac{1}{10^{\ell-1}}\right) = m_0 + \frac{9}{10}\left(\frac{1-\frac{1}{10^\ell}}{1-\frac{1}{10}}\right) <$$

$$m_0 + \frac{9}{10}\left(\frac{1}{1-\frac{1}{10}}\right) = m_0 + 1$$

Therefore $\sup U$ exists in R. The nontrivial claim is that we can take $x = \sup U$, and then $f(x) = m_0.m_1\dots$. Indeed, here is the sketch of the argument. Assume the contrary that $f(x) = n_0.n_1n_2\dots$ and $n_0.n_1n_2\dots \neq m_0.m_1\dots$. This means that there exists $k \geq 1$ such that $n_0 = m_0, n_1 = m_1, \dots, n_{k-1} = m_{k-1}$, $m_k \neq n_k$. consider the case $n_k < m_k$ (the case $n_k > m_k$ is similar). Let V be the corresponding set of numbers for the element $n_0.n_1n_2\dots$, i.e.,

$$V = \left\{n_0, \quad n_0 + \frac{n_1}{10}, \quad n_0 + \frac{n_1}{10} + \frac{n_2}{10^2}, \dots\right\}.$$

It suffices to show that $\sup V < \sup U$. Let us use the notation for "finite" sequences of $\mathbb{W}$, i.e.,

$$h_0.h_1\dots h_n = h_0 + \frac{h_1}{10} + \dots + \frac{h_n}{10^n}$$

Next, we will show that

$$\sup V < n_0.n_1n_2\dots n_{k-1}(n_k+1) \leq n_0.n_1n_2\dots n_{k-1}m_k\dots \leq \sup U. \tag{7.1}$$

Indeed, let $\ell > k$ be such that $n_\ell < 9$. Then for any $p \geq 0$ we have

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \dots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} + \dots + \frac{n_\ell}{10^\ell} + \frac{n_{\ell+1}}{10^{\ell+1}} + \dots + \frac{n_{\ell+p}}{10^{\ell+p}} \leq$$

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \dots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} + \dots + \frac{n_\ell}{10^\ell} + \frac{9}{10^{\ell+1}} + \dots + \frac{9}{10^{\ell+p}} \leq$$

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \dots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} + \dots + \frac{n_\ell + 1}{10^\ell} \leq$$

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \dots + \frac{n_k}{10^k} + \frac{9}{10^{k+1}} + \dots + \frac{9}{10^\ell} =$$

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \dots + \frac{n_k + 1}{10^k} + \left(\frac{9}{10^{k+1}} + \dots + \frac{9}{10^\ell} - \frac{1}{10^k}\right) =$$

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \dots + \frac{n_k + 1}{10^k} - \varepsilon$$

where $\varepsilon = -\left(\frac{9}{10^{k+1}} + \dots + \frac{9}{10^\ell} - \frac{1}{10^k}\right) > 0$. (notice that if we would include infinite tails of 9's then ε would not be separated from zero)

Thus we have proved that $\forall v \in V$ we have

$$v \leq n_0.n_1n_2\dots n_{k-1}(n_k+1) - \varepsilon < n_0.n_1n_2\dots n_{k-1}(n_k+1)$$

In particular

$$\sup V \leq n_0.n_1n_2\dots n_{k-1}(n_k+1) - \varepsilon < n_0.n_1n_2\dots n_{k-1}(n_k+1)$$

Since $m_k \geq n_k + 1$ we have

$$\sup U \geq m_0.m_1\dots m_{k-1}m_k0 \geq m_0.m_1\dots m_{k-1}(n_k+1)$$

which proves the inequality (7.1).

Exercise 1. *What happens when $n_k > m_k$?*

Thus we have identified in 1-1, onto way R_+ and $\mathbb{W} \setminus B$.

The elements of $\mathbb{W} \setminus B$ are called "decimal expansions" of real numbers x, and we just saw that to "identify" all cuts we do not need all possible elements from $\mathbb{W}$.

However it is convenient to say that $0.999\dots$ and $1.000\dots$ correspond to one and the same cut and to identify these two different elements to be "equivalent". We say that $u, v \in \mathbb{W}$ are equivalent if the corresponding set of numbers E_v and E_u have the same supremum, i..e, $\sup E_v = \sup E_u$.

Problem 7.3. *Show that $\mathbb{W} \setminus B$ is uncountable.*

Proof. First show that B is countable. Then show that $\mathbb{W}$ is uncountable by using Cantor's diagonal process (or use the fact that $\mathbb{W}$ contains all infinite sequences 01000110011... with coordinates 0 and 1, and use a theorem that we have proved in the class). Then show that if $\mathbb{W}$ is uncountable and B is countable then $\mathbb{W} \setminus B$ is uncountable. □

Thus we obtain that the set R_+ is uncountable.