PROBLEM SET II (DUE NOV. 9, 2018)

1. Problem 0

Show that intersection of countable number of k-cells, say $\{I_n\}_{n\geq 1}$ is nonempty if $I_{n+1} \subset I_n$ for any $n \geq 1$.

Solution See Rudin.

2. Problem 1

Show that there can be only finite or countable number of pairwise disjoint symbol "8"'s drawn on the plane.

Solution Indeed, to any symbol "8" drawn on the plane choose any two points A and B on the plane with rational coordinates such that A, and B are inside two different circles of 8. Next, show that for two disjoint symbol 8's the pair of points A, B and A', B' will be different, i.e., if $A = A'$ and $B = B'$ then the corresponding symbol 8's will intersect.

This gives a map f from the pairwise disjoint set of 8's drawn on the plane to a subset of $Q^2 \times Q^2$ which is 1-1. The latter means that 8's can be at most countable.

3. Problem 2

Let $\{0,1\}^{\mathbb{N}}$ be the Cantor group, i.e., all possible sequences of the type $(0,0,1,1,0,\ldots)$ consisting of 0's and 1's. Let $\mathcal{P}(\mathbb{N})$ denote the collection of all subsets of natural numbers $\mathbb{N} = \{1,2,\ldots\}$. Construct $1-1$, onto map $f : \{0,1\}^{\mathbb{N}} \mapsto \mathcal{P}(\mathbb{N})$.

Solution: here is the construction: to the point $(1,1,1,\ldots,)$, i.e., all coordinates are 1 we will associate $\{1,2,3,\ldots,\}$ the whole set. Next, if we have somewhere zero then we will remove the corresponding element written on the same position from $\{1,2,3,\ldots,\}$. For example

$$
\begin{aligned}
&(1,1,1,\ldots,) \mapsto \{1,2,3,\ldots,\}\\
&(0,1,1,\ldots,) \mapsto \{2,3,\ldots,\};\\
&(1,0,1,\ldots,) \mapsto \{1,3,\ldots,\};\\
&\ldots\\
&(0,1,1,0,0\ldots,) \mapsto \{2,3\}; \quad \text{etc.}
\end{aligned}
$$

Clearly each point of $\{0,1\}^{\mathbb{N}}$ will define a subset of $\mathbb{N}$. The map is 1-1 and onto.

4. Problem 3

I suggest to solve **all** problems listed in the exercise section of Chapter 2. Here are some of them: 2, 5, 8, 9, 10 (very strange problem), 11, 13, 14, 17 (looks like an interesting problem), 21(c), 22, 23, 24, 25, (these 23, 24, 25, 26 are together, in some literature people define compact sets by saying that every infinite subset has a limit point in it) 26, 29, 30 Chapter 2.

On our midterm exam there will be 5 problems: 4 of them will be from Rudin.

5. Bonus problem 1

Given an integer $n \geq 1$, suppose we want to find n different vectors $v_1, v_2, \ldots, v_n$ in $\mathbb{R}^n$ with coordinates $+1$ or -1 such that the inner product of any two of them is zero. Can one solve this problem when $n = 6$? When $n = 8$? Notice that when $n = 2$ one can take $v_1 = (1, 1)$ and $v_2 = (-1, 1)$. Then clearly $v_1 \cdot v_2 = 0$.

Solution

When $n = 6$ we cannot solve the problem. Indeed, assume yes, and we found $v_1, v_2, \ldots, v_6$ in $\mathbb{R}^6$ with coordinates ± 1 such that $\langle v_j, v_i \rangle = 0$ for any $i \neq j$. Let us write these vectors as rows in the following way

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix}$$

We will see a following list

$$\begin{pmatrix} 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

where the j'th row of the list represents the vector v_j. Notice that if we switch any two columns then the resulting new vectors $v_1', v_2', v_3', v_4', v_5', v_6'$ corresponding to the rows of new list still will satisfy the property that the inner product v_j' and v_i' is zero provided that $i \neq j$. Also notice that the same is true if we would switch the rows of the list. The same is true if we multiply by -1 all elements of any given column. So the latter means that we can switch the signs of any given column. Let us look at the first row: $(1 - 1 \quad 1 - 1 - 1 \quad 1)$. Let us make all -1 to be $+1$ by multiplying the corresponding column by -1. For example, if we multiply the second column by -1 we would get a new list

$$\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

Similarly the fourth and fifth column

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 \end{pmatrix}$$

So what we achieved is that in the first row we made everybody to be $+1$. Since the inner product of the first and the second row is zero, it means that in the second row we must have exactly $\frac{6}{2} = 3$ negative 1's. Let us switch the columns, so that to put all negative ones to the left side, for example like this

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

Next arguing in a similar way in the third row we must have also $6/2 = 3$ negative ones. By switching the last remaining $6/2$ columns we can achieve in the third row that in the last $6/2$ numbers of the third row there must be half negative ones and half positive 1's, and also in the first 3 numbers of the third row (WHY?), but $6/4$ is not an integer so this is not possible.

The construction for $n = 8$ can be done as follows: let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then consider

$$B = \begin{pmatrix} A & A \\ A & -A \end{pmatrix}$$

and finally take

$$C = \begin{pmatrix} B & B \\ B & -B \end{pmatrix}$$

Show that the vectors $v_1, \ldots, v_8$ corresponding to the rows of this new obtained list C do satisfy the property that their inner product is zero.

6. Bonus problem 2

Let $n \geq 1$ be a fixed integer. Suppose we are given on the plane n red vertices and n blue vertices. Suppose there are edges joining red and blue vertex, and we do not know how many edges are there but there is the following constraint: for **any** $k \leq n$, and **any** k-subset of red vertices (call it A) there exist at least k blue vertices (call it B) such that for any vertex $b \in B$ there is an edge joining b to a vertex $a \in A$. Does this constraint imply that we can split red and blue vertices in n pairs of vertices (red,blue) such that vertices from each pair are joined by an edge (each vertex can be only in one pair).

Solution

Proof is by induction on n. If $n = 1$ there is nothing to prove. Assume the claim is true for $k = 1, 2, \ldots, n-1$, and now we want to prove for $k = n$.

Step 1: Take any m subset A of red vertices where $1 \leq m < n$. Suppose we can find **at most** m blue vertices B with the property described in the problem, i.e., any $b \in B$ is joined to some element of A. Then notice that no vertex from B^c (the complement of B) is joined to a vertex of A otherwise we would've find more than m blue vertices of type B. Next notice that for any k subset of A (here $k \leq m$), call it A', there exists at least k vertices in B, call it B' such that any vertex of B' is joined to a vertex of A'. Hence, we can use the induction hypothesis the sets A and B. Next we want to use the induction hypothesis for A^c and B^c. But to do so we should verify that they satisfy the assumption that for any ℓ subset of red vertices in A^c, call it A'', there exists at least ℓ subset of blue vertices in B^c call it B'' such that any vertex of B'' is joined to some vertex of A''. Indeed, if this number of blue vertices is strictly less than ℓ, say $p < \ell$ then consider the set $A'' \cup A$ which has $m + \ell$ red vertices then we can find only $m + p$ blue vertices with the property described in the problem which contradicts to the assumption of the problem. Therefore we can use induction hypothesis to A^c and B^c.

Step 2: In what follows we can assume that for any m subset A of red vertices where $1 \leq m < n$ we can always find **at least** $m + 1$ blue vertices B with the property described in the problem (otherwise we are in step 1 which is solved). In this case take any blue vertex such that it is joined to some red vertex. Call such blue vertex b_0 and its joined red vertex a_0. Clearly they exist. In this case let us remove from the set of red vertices a_0, and from the set of blue vertices b_0 (they are already joined by an edge and let us forget about them). Now we are left with $n-1$ red vertices and $n-1$ blue vertices. The claim is that this new sets satisfy the property described in the problem: for any m subset of red vertices, call it A', there exists at least m subset of blue vertices, call it B' such that any $b \in B'$ is joined to some $a \in A'$. Indeed, with b_0 there were

at least $m+1$ blue vertices, and now by removing b_0 definitely there will be **at least** m blue vertices. Thus we can apply the induction hypothesis to $n-1$ red and $n-1$ blue vertices.

7. Why cuts (real numbers) are uncountable?

Let us construct 1-1 map f from the set R_+ of all nonnegative cuts $x \geq 0$ to the set of all objects of the form

$$n_0.n_1n_2n_3\ldots,$$

where $n_0 \geq 0$ is any integer and $n_1.n_2,\ldots$ can be nonnegative integers taking values $0,1,2,\ldots,9$.

Definition. *All such sequences we denote by* $\mathbb{W}$*.*

Also notice that we can consider negative cuts as well just by putting negative sign everywhere.

Here starts the construction.

Take any $x \in R_+$, $x \geq 0$. Let $n_0 \geq 0$ be the largest integer such that $n_0 \leq x$. Now let $n_1 \geq 0$ be the largest integer such that $n_0 + \frac{n_1}{10} \leq x$. Notice that $0 \leq n_1 \leq 9$ (why n_1 cannot be greater than 9?). And we iterate this process: suppose $n_0, n_1, \ldots, n_k$ are already constructed then we define $n_{k+1} \geq 0$ to be the largest integer such that

$$n_0 + \frac{n_1}{10} + \ldots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \leq x.$$

Thus we have a map $x \mapsto n_0.n_1n_2\ldots$, i.e., $f : R_+ \mapsto \mathbb{W}$.

Why is f one-to-one?

For this it would be enough to show that $x = \sup E$ where

$$E = \left\{ n_0 + \frac{n_1}{10} + \ldots + \frac{n_k}{10^k} \right\}_{k\geq 0}.$$

Indeed, this would mean that if there are two reals $x, y \geq 0$, $x \neq y$ such that their image gives one and the same object $n_0.n_1n_2\ldots$, then $x = \sup E$, and $y = \sup E$ which would be a contradiction (since $\sup E$ is unique). Thus it suffices to verify the property that $x = \sup E$. Notice that E is nonempty, and E is upper bounded (for any $e \in E$ we have $e \leq x$). Therefore $\sup E$ exists in R. Denote $t \stackrel{\text{def}}{=} \sup E$. Next we show that $t = x$. It is clear that $x \geq t$ since x is an upper bound for E and t is the least upper bound. Next, assume the contrary that $t < x$. Set $\varepsilon = x - t > 0$. Let $\ell \geq 0$ be such that $\frac{1}{10^\ell} < \frac{\varepsilon}{2}$. Clearly

$$n_0 + \frac{n_1}{10} + \ldots + \frac{n_\ell}{10^\ell} \leq t$$

Therefore

$$n_0 + \frac{n_1}{10} + \ldots + \frac{n_\ell + 1}{10^\ell} \leq t + \frac{1}{10^\ell} < t + \varepsilon = x$$

But this means that n_ℓ was not the largest nonnegative integer such that

$$n_0 + \frac{n_1}{10} + \ldots + \frac{n_\ell}{10^\ell} \leq x.$$

A contradiction. Thus $x = \sup E$.

Is f onto? The answer is NO.

Indeed, consider the following sequence $0.999\ldots$. And show that one cannot find a cut $x \in R_+$ such that $f(x) = 0.999\ldots$. The "right" candidate should be $x = 1$ but for this one we have $f(1) = 1.000\ldots$. Thus we see that there are "bad" elements in $\mathbb{W}$. Let us call an object $n_0.n_1n_2\ldots \in \mathbb{W}$ bad if starting at some point $k \geq 1$ we have $9 = n_k = n_{k+1} = n_{k+2} = \ldots$. In other words these are those sequences who have a "tail" consisting only by 9's. Let us denote the set of all bad elements of $\mathbb{W}$ by B.

Problem 7.1. *Show that* $f : R_+ \mapsto \mathbb{W} \setminus B$ *is 1-1, and onto.*

Let us split the solution of this problem into several steps. To show that f is 1-1 it will be enough to prove the following lemma

Lemma 7.1. *For each $b \in B$ no $x \in R$ satisfies $f(x) = b$.*

Proof. Here is the sketch: suppose for some $b \in B$ there is x such that $f(x) = b$. We know that starting at some point b has the tail of $9's$. If $b = n_0.999\ldots$ show that the supremum of the set of numbers

$$E = \left\{ n_0, \quad n_0 + \frac{9}{10}, \quad n_0 + \frac{9}{10} + \frac{9}{10^2}, \ldots \right\}$$

is $n_0 + 1 = x$. Indeed, clearly $n_0 + 1$ is un upper bound. It remains to show that it is the least upper bound (Exercise). On the other hand $f(x) = f(n_0 + 1) = (n_0 + 1).000\ldots \neq n_0.999\ldots$.

Now, if

$$b = n_0.n_1n_2\ldots n_k9999\ldots$$

where $n_k < 9$ and $k \geq 1$, then show that the supremum of the corresponding set of E is $b = n_0.n_1n_2\ldots(n_k + 1)000\ldots$. Again this is left as an exercise. □

The lemma together with the previous reasoning implies that the map $f : R_+ \mapsto \mathbb{W} \setminus B$ is one-to-one

Finally it remains to solve the following problem.

Problem 7.2. *Show that $f : R_+ \mapsto \mathbb{W} \setminus B$ is onto.*

Here we need to show that given any object $m_0.m_1m_2\ldots$ (here $m_0 \geq 0$ is any integer and $m_1, m_2, \ldots$ take values $0, 1, 2\ldots, 9$) we can find a nonnegative real x such that $f(x) = m_0.m_1m_2$. Indeed, let U to be the set of all numbers $m_0 + \frac{m_1}{10} + \ldots + \frac{m_k}{10^k}$ for all $k \geq 0$. Then U is nonempty, and it is bounded from above. Indeed, let us show that for any $e \in U$ we have $e \leq m_0 + 1$. We have

$$e = m_0 + \frac{m_1}{10} + \ldots + \frac{m_\ell}{10^\ell}$$

for some $\ell \geq 0$. If $\ell = 0$ there is nothing to prove. Assume $\ell \geq 1$. Then

$$e \leq m_0 + \frac{9}{10}\left(1 + \ldots + \frac{1}{10^{\ell-1}}\right) = m_0 + \frac{9}{10}\left(\frac{1 - \frac{1}{10^\ell}}{1 - \frac{1}{10}}\right) <$$
$$m_0 + \frac{9}{10}\left(\frac{1}{1 - \frac{1}{10}}\right) = m_0 + 1$$

Therefore $\sup U$ exists in R. The nontrivial claim is that we can take $x = \sup U$, and then $f(x) = m_0.m_1\ldots$. Indeed, here is the sketch of the argument. Assume the contrary that $f(x) = n_0.n_1n_2\ldots$ and $n_0.n_1n_2\ldots \neq m_0.m_1\ldots$. This means that there exists $k \geq 1$ such that $n_0 = m_0, n_1 = m_1, \ldots, n_{k-1} = m_{k-1}$, $m_k \neq n_k$. consider the case $n_k < m_k$ (the case $n_k > m_k$ is similar). Let V be the corresponding set of numbers for the element $n_0.n_1n_2\ldots$, i.e.,

$$V = \left\{ n_0, \quad n_0 + \frac{n_1}{10}, \quad n_0 + \frac{n_1}{10} + \frac{n_2}{10^2}, \ldots \right\}.$$

It suffices to show that $\sup V < \sup U$. Let us use the notation for "finite" sequences of $\mathbb{W}$, i.e.,

$$h_0.h_1\ldots h_n = h_0 + \frac{h_1}{10} + \ldots + \frac{h_n}{10^n}$$

Next, we will show that

$$\sup V < n_0.n_1n_2\ldots n_{k-1}(n_k + 1) \leq n_0.n_1n_2\ldots n_{k-1}m_k\ldots \leq \sup U. \tag{7.1}$$

Indeed, let $\ell > k$ be such that $n_\ell < 9$. Then for any $p \geq 0$ we have

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \ldots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} + \ldots + \frac{n_\ell}{10^\ell} + \frac{n_{\ell+1}}{10^{\ell+1}} + \ldots + \frac{n_{\ell+p}}{10^{\ell+p}} \leq$$
$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \ldots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} + \ldots + \frac{n_\ell}{10^\ell} + \frac{9}{10^{\ell+1}} + \ldots + \frac{9}{10^{\ell+p}} \leq$$
$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \ldots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} + \ldots + \frac{n_\ell + 1}{10^\ell} \leq$$
$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \ldots + \frac{n_k}{10^k} + \frac{9}{10^{k+1}} + \ldots + \frac{9}{10^\ell} =$$
$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \ldots + \frac{n_k + 1}{10^k} + \left(\frac{9}{10^{k+1}} + \ldots + \frac{9}{10^\ell} - \frac{1}{10^k}\right) =$$
$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \ldots + \frac{n_k + 1}{10^k} - \varepsilon$$

where $\varepsilon = -\left(\frac{9}{10^{k+1}} + \ldots + \frac{9}{10^\ell} - \frac{1}{10^k}\right) > 0$. (notice that if we would include infinite tails of 9's then ε would not be separated from zero)

Thus we have proved that $\forall v \in V$ we have

$$v \leq n_0.n_1n_2\ldots n_{k-1}(n_k+1) - \varepsilon < n_0.n_1n_2\ldots n_{k-1}(n_k+1)$$

In particular

$$\sup V \leq n_0.n_1n_2\ldots n_{k-1}(n_k+1) - \varepsilon < n_0.n_1n_2\ldots n_{k-1}(n_k+1)$$

Since $m_k \geq n_k + 1$ we have

$$\sup U \geq m_0.m_1\ldots m_{k-1}m_k0 \geq m_0.m_1\ldots m_{k-1}(n_k+1)$$

which proves the inequality (7.1).

Exercise 1. *What happens when $n_k > m_k$?*

Thus we have identified in 1-1, onto way R_+ and $\mathbb{W} \setminus B$.

The elements of $\mathbb{W} \setminus B$ are called "decimal expansions" of real numbers x, and we just saw that to "identify" all cuts we do not need all possible elements from $\mathbb{W}$.

However it is convenient to say that $0.999\ldots$ and $1.000\ldots$ correspond to one and the same cut and to identify these two different elements to be "equivalent". We say that $u, v \in \mathbb{W}$ are equivalent if the corresponding set of numbers E_v and E_u have the same supremum, i..e, $\sup E_v = \sup E_u$.

Problem 7.3. *Show that $\mathbb{W} \setminus B$ is uncountable.*

Proof. First show that B is countable. Then show that $\mathbb{W}$ is uncountable by using Cantor's diagonal process (or use the fact that $\mathbb{W}$ contains all infinite sequences 01000110011... with coordinates 0 and 1, and use a theorem that we have proved in the class). Then show that if $\mathbb{W}$ is uncountable and B is countable then $\mathbb{W} \setminus B$ is uncountable. □

Thus we obtain that the set R_+ is uncountable.