PROBLEM SET III (DUE FEB. 3, 2019)

1. Problem 1

Rudin, Chapter 7, Problem 6

2. Problem 2

Rudin, Chapter 7, Problem 7

3. Problem 3

Prove that if the series $\sum_{n\geq 0} a_n$ converges then

$$\lim_{t\to 1-0} \sum_{n\geq 0} a_n t^n = \sum_{n\geq 0} a_n.$$

By the symbol $\lim_{t\to 1-0}$ we mean left-hand limit, i.e., when t approaches to 1 from the left side.

Hint: Without loss of generality assume that $\sum_{n\geq 0} a_n = 0$, and use the identity $\sum_{n\geq 0} a_n t^n = (1-t)\sum_{n\geq 0} S_n t^n$ where $S_n := a_0 + \ldots + a_n$.

4. Problem 4

$f : \mathbb{R} \to \mathbb{R}$ is said to be piece-wise linear function if there are finite number of points $c_0 < c_1 < \ldots < c_n$ such that f is linear on each intervals $(-\infty, c_0), (c_1, c_2), \ldots, (c_{n-1}, c_n), (c_n, \infty)$. Here linear means that $f(x) = Ax + B$ for some constants $A, B \in \mathbb{R}$.

Assume $f : \mathbb{R} \to \mathbb{R}$ is piecewise linear convex function. Show that one can write f as a linear combination (with nonnegative coefficients) of a linear function, and functions of the form $\varphi(x) = \max\{0, c - x\}$ for some $c \in \mathbb{R}$. In other words there exist constants $A, B, c_1, \ldots, c_m \in \mathbb{R}$, $\alpha_0, \alpha_1, \ldots, \alpha_m \geq 0$ such that

$$f(x) = \alpha_0(A + Bx) + \sum_{j=1}^{m} \alpha_j \max\{0, c_j - x\}$$

Remark 4.1. *Essentially the problem says that any convex function (not necessarily piecewise linear) can be very well approximated by a linear combination with nonnegative coefficients of such simple functions as* $Ax + B$ *and* $\max\{0, c - x\}$.

As an application of this exercise it follows that $\int_a^b f(x)\varphi(x)dx$ is nonnegative for **all** convex functions φ on $[a, b]$ **if and only if** $\int_a^b f(x)dx = 0$, $\int_a^b f(x)xdx = 0$, and $\int_a^c (c-x)f(x)dx \geq 0$ for all $c \in (a, b)$.