PROBLEM SET III (DUE NOV. 23, 2018)

1. Problem 0

Rudin, Chapter 3, Exercise: 2, 5, 6 (a,b,c), 7, 8, 9 (a,c,d), 10, 11, 16, 19.

2. Problem 1

Let $a > 0$. Define the sequence

$$x_n = \sqrt{a + \sqrt{a + \ldots + \sqrt{a}}}$$

(here we have n square roots). Show that $\lim_{n\to\infty} x_n = \frac{1}{2}(1 + \sqrt{1+4a})$.

3. Problem 2

Show that for any sequence $\{x_n\}_{n\geq 1}$ of positive real numbers we have

$$\limsup_{n\to\infty} \left(\frac{x_1 + x_{n+1}}{x_n}\right)^n \geq e.$$

4. Problem 3*

Show that if the series $\sum_{n=1}^{\infty} a_n$ converges then there exists a sequence $c_1 \leq c_2 \leq c_3 \ldots$ such that $c_n \to \infty$ and the series $\sum_{n=1}^{\infty} a_n c_n$ converges.

5. Problem 4*

Let the sequence of real numbers $\{x_n\}_{n\geq 1}$ be such that $x_{n+m} \leq x_n + x_m$ for all integers $n, m \geq 1$. Show that $\frac{x_n}{n} \to \inf_{m\geq 1} \frac{x_m}{m}$