PROBLEM SET III (DUE NOV. 23, 2018)

1. Problem 0

Rudin, Chapter 3, Exercise: 2, 5, 6 (a,b,c), 7, 8, 9 (a,c,d), 10, 11, 16, 19.

2. Problem 1

Let $a > 0$. Define the sequence

$$x_n = \sqrt{a + \sqrt{a + \ldots + \sqrt{a}}}$$

(here we have n square roots). Show that $\lim_{n\to\infty} x_n = \frac{1}{2}(1 + \sqrt{1+4a})$.

Solution: Notice that $x_{n+1} \geq x_n$ i.e., the sequence is nondecreasing. Indeed, the case $\sqrt{a+\sqrt{a}} > \sqrt{a}$ easily follows by squaring both sides. And the general case

$$\sqrt{a + \underbrace{\sqrt{a + ...\sqrt{a}}}_{n}} \geq \underbrace{\sqrt{a + ...\sqrt{a}}}_{n} \tag{2.1}$$

follows by induction: we square both sides of (2.1) and we cancel a, then the inequality is equivalent to the following one

$$\underbrace{\sqrt{a + ...\sqrt{a}}}_{n} \geq \underbrace{\sqrt{a + ...\sqrt{a}}}_{n-1}.$$

which is true by induction hypothesis.

Also notice that $x_n \leq a + 1$. Indeed (by induction): the case $x_1 = \sqrt{a} \leq a+1$ is obvious. For the general case notice that

$$x_{n+1} = \sqrt{a + x_n} \overset{\text{induction}}{\leq} \sqrt{a + a + 1} \leq a + 1.$$

The latter inequality holds by squaring both sides of the inequality.

Since $x_{n+1} \geq x_n$ and $x_n \leq a+1$ for all $n \geq 1$ it follows that the limit $\lim_{n\to\infty} x_n = L > 0$ exists. Since $\lim x_{n+1} = \lim x_n$ we have

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{a + x_n} = \sqrt{a + \lim_{n\to\infty} x_n} = \sqrt{a+L}$$

The solution of the equation $L = \sqrt{a+L}$ is precisely $L = \frac{1}{2}(1+\sqrt{1+4a})$.

3. Problem 2

Show that for any sequence $\{x_n\}_{n\geq 1}$ of positive real numbers we have

$$\limsup_{n\to\infty} \left(\frac{x_1 + x_{n+1}}{x_n}\right)^n \geq e.$$

Solution:

Assume the contrary, i.e.,

$$\limsup_{n\to\infty} \left(\frac{x_1 + x_{n+1}}{x_n}\right)^n < e.$$

Since $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ it follows that starting from some (sufficiently large) $N > 0$, we have

$$\left(\frac{x_1 + x_{n+1}}{x_n}\right)^n \leq \left(1 + \frac{1}{n}\right)^n \quad \text{for all} \quad n \geq N.$$

The latter implies that $\frac{x_1+x_{n+1}}{x_n} \leq 1 + \frac{1}{n}$ which can be rewritten as follows

$$\frac{x_1}{n+1} \leq \frac{x_n}{n} - \frac{x_{n+1}}{n+1} \quad \text{for all} \quad n \geq N. \tag{3.1}$$

But this is impossible because choose M very very large (much larger than N). Then we can write

$$x_1 \geq \frac{x_1}{1} - \frac{x_{M+1}}{M+1} \overset{\text{telesc.sum}}{=} \sum_{k=1}^{M}\left(\frac{x_k}{k} - \frac{x_{k+1}}{k+1}\right) = \sum_{k=1}^{N-1}\left(\frac{x_k}{k} - \frac{x_{k+1}}{k+1}\right) + \sum_{k=N}^{M}\left(\frac{x_k}{k} - \frac{x_{k+1}}{k+1}\right) \overset{(3.1)}{\geq}$$
$$\sum_{k=1}^{N-1}\left(\frac{x_k}{k} - \frac{x_{k+1}}{k+1}\right) + x_1 \sum_{k=N}^{M} \frac{1}{k+1}$$

Now notice that $\sum_{k=1}^{N-1}\left(\frac{x_k}{k} - \frac{x_{k+1}}{k+1}\right)$ is a fixed number (independent of M). On the other hand choosing M large enough we can make the second term, i.e., $x_1 \sum_{k=N}^{M} \frac{1}{k+1}$ as large as we wish (because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges). Thus we get that

$$x_1 \geq (\text{a fixed number}) + x_1 \cdot (\text{a large number going to infinity as M goes to inifinity}).$$

Choosing M large enough we come to a contradiction.

4. Bonus problem 3*

Show that if the series $\sum_{n=1}^{\infty} a_n$ converges then there exists a sequence $c_1 \leq c_2 \leq c_3 \ldots$ such that $c_n \to \infty$ and the series $\sum_{n=1}^{\infty} a_n c_n$ converges.

Solution: We will use the fact that partials sums form Cauchy sequence: since the series $\sum_{n=1}^{\infty} a_n$ converges then by choosing N large enough we can make the difference of partial sums $|s_p - s_q|$ as small as we wish provided that $q \geq p \geq N$.

Indeed, pick any sequence $\{\varepsilon_j\}_{j\geq 1}$ going to zero "sufficiently fast", for example $\varepsilon_j = \frac{1}{j^{10}}$ for $j \geq 1$. Next, choose $N_1 > 0$ so that

$$\left|\sum_{p\leq k\leq q} a_k\right| < \varepsilon_1 \quad \text{for all} \quad p \geq q \geq N_1,$$

Similarly choose $N_2 > N_1$ so that $\left|\sum_{p\leq k\leq q} a_k\right| < \varepsilon_2$ for all $p \geq q \geq N_2$ etc.

Define $c_\ell = j$ if $N_j \leq \ell < N_{j+1}$ for all $j \geq 1$, and $c_\ell = 0$ if $1 \leq \ell < N_1$. Clearly $\{c_\ell\}_{\ell\geq 1}$ is nondecreasing going to infinity as $\ell \to \infty$. Also notice that if $s > t > N$ for some large $N > 0$, say if $s \in [N_u, N_{u+1})$ and $t \in [N_v, N_{v+1})$ where $u \geq v$ and v is large enough then we have

$$\left|\sum_{t\leq k\leq s} a_k c_k\right| \leq u \left|\sum_{t\leq k<N_{u+1}} a_k\right| + (u+1)\left|\sum_{N_{u+1}\leq k<N_{u+2}} a_k\right| + \ldots + v \left|\sum_{N_v\leq k\leq s} a_k\right| \leq$$
$$\frac{u}{u^{10}} + \frac{u+1}{(u+1)^{10}} + \ldots + \frac{v}{v^{10}}$$

which goes to zero as N (i.e., u) goes to infinity. Therefore $\sum a_k c_k$ converges.

5. Bonus problem 4*

Let the sequence of real numbers $\{x_n\}_{n\geq 1}$ be such that $x_{n+m} \leq x_n + x_m$ for all integers $n, m \geq 1$. Show that $\frac{x_n}{n} \to \inf_{m\geq 1} \frac{x_m}{m}$

Solution: Let $\ell = \inf_{m\geq 1} \frac{x_m}{m}$. It is enough to show that for any number $s > \ell$ we have

$$\ell \leq \liminf_{n\to\infty} \frac{x_n}{n} \leq \limsup_{n\to\infty} \frac{x_n}{n} \leq s. \tag{5.1}$$

This would imply that $\lim_{n\to\infty} \frac{x_n}{n} = \ell$. To verify (5.1), let $m \geq 1$ be such that $\frac{x_m}{m} < s$. Pick any integer $n \geq 1$. We can write it as $n = mk + r$ where $0 \leq r \leq m-1$, and $k = k(n)$, $r = r(n)$ are nonnegative integers. Then

$$\frac{x_n}{n} = \frac{x_{mk+r}}{n} \leq \frac{kx_m}{n} + \frac{x_r}{n} < \frac{km}{n}s + \frac{x_r}{n} \tag{5.2}$$

Now when n goes to infinity clearly $\frac{x_r}{n} \to 0$ because always $x_r \in \{x_1, x_2, \ldots, x_{m-1}\}$ which are fixed numbers (independent of n). Also $\frac{km}{n}$ goes to 1 as $n \to \infty$ because $1 = \frac{km}{n} + \frac{r}{n}$ and $\frac{r}{n}$ goes to zero. Thus the limit of the right hand side of (5.2) is s. This implies that $\limsup_{n\to\infty} \frac{x_n}{n} \leq s$.