PROBLEM SET 4 (DUE FEB. 10, 2019)

1. Problem 1

Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Define

$$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} \binom{n}{k}.$$

Show that polynomials B_n converge uniformly to f on $[0,1]$ as $n \to \infty$.

Hint: notice that since $1 = \sum_{0 \le k \le n} x^k (1-x)^{n-k} \binom{n}{k}$ it suffices to bound uniformly in x the sum

$$|B_n(x) - f(x)| \le \sum_{0 \le k \le n} \left| f\left(\frac{k}{n}\right) - f(x) \right| x^k (1-x)^{n-k} \binom{n}{k}.$$

If k is such that $\left|\frac{k}{n} - x\right| \le \delta$ then we can use uniform continuity of f to bound these terms. If $\left|\frac{k}{n} - x\right| > \delta$, then we can estimate $\left|f\left(\frac{k}{n}\right) - f(x)\right| \le 2M$, where M is the maximum of f on $[0,1]$. Therefore it remains to show that the sum

$$\sum_{k : \left|\frac{k}{n} - x\right| > \delta} x^k (1-x)^{n-k} \binom{n}{k}$$

is small as n becomes large. The question is how to show it?

Remark 1.1. *The intuition behind the polynomials $B_n(x)$ (which are called Bernstein polynomials) is "probabilistic" and it comes from the fact that $B_n(x) = \mathbb{E} f\left(\frac{\xi_1 + \ldots + \xi_n}{n}\right)$ where $\xi_1, \ldots, \xi_n$ are independent identically distributed Bernoulli 0,1 random variables, i.e., $\xi_j = 1$ with probability x and $\xi_j = 0$ with probability $(1-x)$. Since $\lim_{n\to\infty} \frac{\xi_1 + \ldots + \xi_n}{n}$ "converges" to its average value $\mathbb{E}\frac{\xi_1 + \ldots + \xi_n}{n} = x$ (by "strong law of large numbers"), then it is "reasonable" to expect that for any continuous function f on $[0,1]$ the quantity $f\left(\frac{\xi_1 + \ldots + \xi_n}{n}\right)$ also "converges" to $f(x)$, and in particular by "Lebesgue dominated convergence theorem" we can pass to the limit under "the integrals", i.e., $\mathbb{E} f\left(\frac{\xi_1 + \ldots + \xi_n}{n}\right)$ "converges" to $f(x)$. All one needs to justify is why the convergence is uniform in x.*

2. Problem 2

Consider the family of nonnegative functions $\{\varphi_k\}_{k \ge 0}$ on the real line $\mathbb{R}$ such that

1. $\int_{\mathbb{R}} \varphi_k(x) dx = 1$ for all $k \ge 0$.
2. For any $\delta > 0$ we have $\lim_{k\to\infty} \int_{|x| \ge \delta} \varphi_k(x) dx = 0$.

Here $\int_{\mathbb{R}} \varphi_k dx = \int_{-\infty}^{\infty} \varphi_k dx$; and $\int_{|x| \ge \delta} \varphi_k = \int_{-\infty}^{-\delta} \varphi_k + \int_{\delta}^{\infty} \varphi_k$. Let f be a uniformly continuous bounded function on $\mathbb{R}$. Define $f * \varphi_k(x) = \int_{\mathbb{R}} f(y) \varphi_k(x-y) dy$. Show that $f * \varphi_k$ converges uniformly to f on $\mathbb{R}$ as $k \to \infty$.

3. Problem 3

Rudin, Chapter 7, problem 14

4. Problem 4

Rudin, Chapter 7, problem 16.

5. Problem 5

Rudin, Chapter 7, problem 25.