PROBLEM SET IV (DUE DEC. 7, 2018)

1. Problem 0

Rudin, Chapter 4, Exercise: 3, 4, 5, 6, 7, 8, 9, 14, 16, 18, 20, 21, 23, 24, 25(a).

2. Bonus problem 1

For a real number $x > 0$ let $\{x\}$ denote its fractional part, i.e., $\{x\} = x - [x]$ where $[x]$ denotes the largest integer smaller than x. Show that for any $x > 1$ we have

$$\sum_{k=1}^{n} \{kx\} \leq \frac{n}{2}x,$$

holds true for all $n \geq 1$.

Solution: Since I have explained the proof of the inequality in the class I will only sketch the idea: it is enough to consider the case $2 \geq x \geq 1$. Taking $x = y + 1$, $y \in (0,1)$, and using the fact that $\{kx\} = \{ky\} = ky - [ky]$ it is enough to show that $\sum_{k=1}^{n}[ky] \geq \frac{n(n+1)y}{2} - \frac{n}{2}(y+1) = \frac{n \cdot ny}{2} - \frac{n \cdot 1}{2}$. To verify the last inequality, i.e., $\frac{n \cdot 1}{2} + \sum_{k=1}^{n}[ky] \geq \frac{n \cdot ny}{2}$ draw a graph of a function $f(s) = sy$ for $s \in [0,n]$ and also draw a square lattice and try see what does this inequality mean (compare areas).

3. Bonus problem 2

Let f be a convex function such that the series $\sum_{k=1}^{\infty} kf(k)$ converges absolutely. Show that

$$\sum_{k=1}^{\infty}(-1)^{k-1}kf(k) \geq 0.$$

Solution Since the series converges absolutely we can rearrange terms and put the parentheses as we wish. For example,

$$\sum_{k=1}^{\infty}(-1)^{k-1}kf(k) =$$

$$f(1) - 2f(2) + 3f(3) - 4f(4) + 5f(5) - 6f(6) + 7f(7) - 8f(8) + 9f(9) - 10f(10) + ... =$$

$$[f(1) - 2f(2) + f(3)] + 2[f(3) - 2f(4) + f(5)] + 3[f(5) - 2f(6) + f(7)] + 4[f(7) - 2f(8) + f(9)] + ...$$

Since each terms is nonnegative by convexity, i.e., $f(a) - 2f(\frac{a+b}{2}) + f(b) \geq 0$ we obtain that the full sum is nonnegative.

4. Bonus problem 3

Let $I \subset \mathbb{R}$ be an interval, and let $f : I \mapsto \mathbb{R}$ be a non-decreasing convex function. If $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are numbers in I such that

$$x_1 \geq x_2 \geq \ldots \geq x_n,$$
$$y_1 \geq y_2 \geq \ldots \geq y_n,$$

and

$$x_1 + \ldots + x_k \geq y_1 + \ldots y_k, \quad \text{for all} \quad 1 \leq k \leq n$$

then

$$f(x_1) + \ldots + f(x_n) \geq f(y_1) + \ldots + f(y_n)$$

Solution Without loss of generality assume that $x_i \neq y_i$ for all $i = 1, \ldots, n$ (how?). Next, set

$$D_j = \frac{f(x_i) - f(y_i)}{x_i - y_i} \quad \text{for all} \quad j = 1, \ldots, n.$$

First show that convexity of f implies that $D_j - D_{j+1} \geq 0$ for all $j = 1, \ldots, n-1$. Next, let $X_j = x_1 + .. + x_j$, and $Y_j = y_1 + .. + y_j$ for all $j = 1, \ldots, n$. Also set $X_0 = Y_0 = 0$. Clearly $X_j - Y_j \geq 0$ for all $j = 1, \ldots, n$.

We can write

$$\begin{aligned}
&f(x_1) - f(y_1) + f(x_2) - f(y_2) + ... + f(x_n) - f(y_n) = \\
&= D_1(x_1 - y_1) + D_2(x_2 - y_2) + ... + D_n(x_n - y_n) = \\
&= D_1(X_1 - X_0 - (Y_1 - Y_0)) + D_2(X_2 - X_1 - (Y_2 - Y_1)) + ... + D_n(X_n - X_{n-1} - (Y_n - Y_{n-1})) = \\
&= D_1(X_1 - Y_1) + [D_2(X_2 - Y_2) - D_2(X_1 - Y_1)] + [D_3(X_3 - Y_3) - D_3(X_2 - Y_2)] + ...+ \\
&+ D_n(X_n - Y_n) - D_n(X_{n-1} - Y_{n-1}) = \\
&(X_1 - Y_1)(D_1 - D_2) + (X_2 - Y_2)(D_2 - D_3) + ... + (X_{n-1} - Y_{n-1})(D_{n-1} - D_n) + D_n(X_n - Y_n)
\end{aligned}$$

All terms are nonnegative. The last term is nonnegative because $D_n \geq 0$ by the fact that f is non-decreasing.

Remark 4.1. *The inequality is called Karamata's inequality. In fact the converse is also true, if the inequality on f holds for all such choices of points $x_1, .., x_n$, and $y_1, ..., y_n$ (and some $n \geq 2$) then f is nondecreasing and convex (exercise).*