PROBLEM SET 6 (DUE FEB. 25, 2019)

Given a positive number n let $\sigma_0(n)$ be the number of the divisors of n, and let $\sigma_1(n)$ be the sum of all divisors of n. For example if $n = 12$ then all divisors are $1, 2, 3, 4, 6, 12$. Therefore $\sigma_0(n) = 6$, and $\sigma_1(n) = 1 + 2 + 3 + 4 + 6 + 12 = 28$.

1. Problem 1

Show that for any $|t| < 1$ we have

$$\sum_{n\geq 1} \frac{t^n}{1-t^n} = \sum_{n\geq 1} \sigma_0(n) t^n$$

2. Problem 2

Show that

$$\sum_{n\geq 1} \frac{nt^n}{1-t^n} = \sum_{n\geq 1} \sigma_1(n) t^n$$

3. Problem 3 ("after twelve years")

Given $x \in \mathbb{R}$ let $[x]$ denote its "floor function", i.e., the greatest integer less than or equal to x. Consider the partial sums

$$S_N(x) = \sum_{n=1}^{N} (-1)^{[nx]}$$

For which rational numbers $x = \frac{p}{q} \in \mathbb{Q}$ we have

$$\sup_{N\geq 1} |S_N(x)| < \infty\,?$$

Remark 3.1. *In fact one can show that if x is irrational then the quantity $\sup_{N\geq 1} |S_N(x)|$ is not bounded! This is a little bit tricky but if time permits I will cover it in the class.*

4. Bonus problem

Let x be an irrational number. Show that $\sup_{N\geq 1} |S_N(x)|$ is bounded if and only if

$$\sup_{n\geq 1} \left| \#\{1 \leq k \leq n \,:\, 0 \leq \{kx\} \leq \frac{1}{2}\} - \frac{n}{2} \right| < \infty$$

where $\{kx\} = kx - [kx]$, and $\#$ denotes the cardinality of the set of those positive integers k, $1 \leq k \leq n$ such that $0 \leq \{kx\} \leq \frac{1}{2}$.

Remark 4.1. *Solutions of the bonus problems are not graded. Nevertheless, I would be happy to see your solution which you could tell me before/after the class or during my office hours.*

5. Bonus problem

Let f be a 1-periodic continuous function on $[0, 1)$. Let $x \in \mathbb{R} \setminus \mathbb{Q}$ be fixed. Assume that

$$\sup_{N\geq 1} \left| \sum_{n=1}^{N} f(nx) \right| < \infty.$$

What can you say about f in terms of its Fourier coefficients?