PROBLEM SET 7 (DUE MAR. 3, 2019)

1. Problem 1

Chapter 8, Exercise 4,5,6,7,8 and 9.

2. Problem 2

We have explained in the class that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

has no zeros on $\mathbb{R}$. The argument was based on the fact that $e^x > 0$ for $x > 0$ and the identity $e^x e^{-x} = 1$ which followed from the binomial identity $(x+y)^n = \sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k}$.

Let us see if we can avoid these identities and give a pure analytic proof of this and a more general fact.

Let $\alpha \in (0,1)$. Show that

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^\alpha} > 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

The interesting case is of course when $x < 0$.

Hints:

1. Show that there exists a constant $c = c(\alpha) > 0$ such that

$$\int_0^1 (1 - t^n) \frac{dt}{t\left(\log \frac{1}{t}\right)^{\alpha+1}} = cn^\alpha$$

holds true for all nonnegative $n = 0, 1, 2, \ldots$.

2. Next, conclude that

$$\int_0^1 (f(x) - f(xt)) \frac{dt}{t\left(\log \frac{1}{t}\right)^{\alpha+1}} = cxf(x) \quad \text{for all} \quad x \in \mathbb{R} \tag{2.1}$$

(explain why one can switch the sum and the integral).

3. Show that the identity (2.1) implies that f cannot have roots at negative real numbers (assume contrary and choose the maximal root).