PROBLEM SET 9 (DUE MAR. 15, 2019)

1. Problem 1

Investigate convergence/divergence of the following integrals
$$\int_1^\infty \cos(x^3 - x)dx, \quad \int_1^\infty \frac{x^q dx}{1 + x^p |\sin(x)|^r}, \quad p, q, r > 0.$$

2. Problem 2

Show that
$$\int_0^{\pi/2} \cos^p(x)dx = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1+p}{2})}{\Gamma(1+\frac{p}{2})}, \ p > 0. \qquad \int_2^\infty \frac{\pi(x)}{x^3 - x}dx = -\frac{1}{2}\ln\left(\frac{6}{\pi^2}\right),$$
where $\pi(x)$ denotes the number of primes not exceeding x (hint: you may want to use the identity $\lim_{n\to\infty}\prod_{k=1}^n \left(1 - \frac{1}{p_k^2}\right) = \frac{6}{\pi^2}$, where $p_1 = 2, p_2 = 3, p_4 = 5$ ... are all primes in increasing order).

3. Problem 3

Find the asymptotic behavior of the following integrals as n goes to infinity.
$$\int_0^\infty \left(\frac{\sin(x)}{x}\right)^n dx; \quad \int_0^2 (1 - 4x + 2x^2)^n \frac{dx}{\sqrt{x}}$$

4. Problem 4

Show that
$$\int_0^x t^x e^{-t} dt \sim \frac{1}{2}\Gamma(1+x) \quad \text{as} \quad x \to \infty.$$
hint: use Laplace's method in problems 3 and 4.

5. Problem 5 (bonus)

Let $f(x) = \sum_{|n|\le N} c_n e^{inx}$. Show that there exists a universal constant $C > 0$ independent of N and f such that
$$\sup_{x\in\mathbb{R}} |f'(x)| \le CN \sup_{x\in\mathbb{R}} |f(x)|$$
hint: construct g such that $f'(x) = f * g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)dy$, and $\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(t)|dt < CN$. Then show that $\sup|f'| = \sup|f * g| \le \sup|f|\frac{1}{2\pi}\int_{-\pi}^{\pi}|g| \le CN\sup|f|$.