PROBLEM SET I (DUE OCT. 26/2018)

1. Problem 1 (Proposition 1.18, Rudin)

The following statements are true in every ordered field

(a) If $x > 0$ then $-x < 0$, and vice versa.
(b) If $x > 0$ and $y < z$ then $xy < xz$.
(c) If $x < 0$ and $y < z$ then $xy > xz$.
(d) If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.
(e) If $0 < x < y$ then $0 < 1/y < 1/x$.

Solution: See Rudin.

2. Problem 2

Let $\alpha \in \mathbb{R}$ be a cut, and let w be a positive rational number. Show that there exists an integer n such that $nw \in \alpha$ but $(n+1)w \notin \alpha$.

Solution: first we show that there exists a positive integer $m > 0$ such that $mw \notin \alpha$. Indeed, assume contrary. Consider the set of all rational cuts $A = \{mw\}_{m \geq 1}$. Then α is an upper bound for this set A. Put $t = \sup A$. Since $w > 0$ it follows that $t - w < t$, i.e., $t - w$ is not an upper bound for A. Hence there exists an element in A, say $m_1 w$ such that $m_1 w > t - w$. The latter implies that $w(m_1 + 1) > t$ which is a contradiction. Thus there exists a positive integer m such that $mw \notin \alpha$.

Next, look at $(m-1)w$. If it happens that $(m-1)w \in \alpha$ then we are done because we can take $n = m - 1$. Otherwise we continuo and look at $(m-2)w$, .. etc. Let us show that for some positive integer k we must have $(m-k)w \notin \alpha$ (the process will stop). Indeed, otherwise consider the set $B = \{pw\}_{p \leq m}$, and notice that $\alpha < pw$ for any integer $p \leq m$, i.e., α is a lower bound. Then arguing similarly as before we come to a contradiction.

3. Problem 3

Let $1^* = \{p \in Q : p < 1\}$. Show that for any cut $\alpha \in R$, $\alpha > 0^*$, there exists a cut $\beta \in R$ such that $\alpha\beta = 1^*$.

Solution: Define $\beta = \{p \in Q, p > 0 : \exists r > 1, \frac{1}{pr} \notin \alpha\} \cup \{p \in Q, p \leq 0\}$. Show that β is a cut. Indeed, β is not empty: take p such that $p \notin \alpha$. Then $1/2p$ is in β. Next, β is not all Q. Indeed, take rational number $q > 0$ and $q \in \alpha$. Then $1/q$ is not in β. Next, β has the property that if $b \in \beta$ then $b + r' \in \beta$ for some $r' > 0$. Indeed, without loss of generality assume that $b > 0$. We know that $\frac{1}{br} \notin \alpha$ for some $r > 1$. If we represent $br = (b+\varepsilon)r'$ for some $r' > 1$ and $\varepsilon > 0$ then we are done because $b + \varepsilon \in \beta$. Indeed, choose $\varepsilon > 0$ such that $\frac{br}{b+\varepsilon} > 1$, for example any $0 < \varepsilon < br - b$ works fine.

Next, we prove $\alpha\beta = 1^*$. This is done in two steps.

Part I: $1^* \subset \alpha\beta$.

Recall that $\alpha\beta$ are those rational numbers p such that $p \leq ab$ for some choices $a \in \alpha$, $b \in \beta$ and $a, b > 0$.

Pick any element $q < 1$, $q > 0$. It is enough to show that $q \in \alpha\beta$ (if $q \leq 0$ there is nothing to prove). Let n be a positive integer (large enough) such that $\frac{n}{n+2} > q$. Clearly such n exists because $q < 1$.

Let $u > 0$ be a positive rational number such that $u \in \alpha$. Consider $w = \frac{u}{n}$. Clearly $w \in \alpha$. There exists a positive integer $m > 0$ such that $mw \in \alpha$ but $(m+1)w \notin \alpha$. Notice that $m \geq n$. Consider $(m+2)w$ which is also not in α. The claim is that $\frac{1}{w(m+2)} \in \beta$. Indeed, take $r = \frac{m+2}{m+1} > 1$, then clearly $\frac{1}{\frac{1}{(m+2)w} \cdot r} = (m+1)w \notin \alpha$.

So we found two elements $\frac{1}{w(m+2)} \in \beta$, and $mw \in \alpha$. Let us show that their product is greater than q. Indeed, we have $\frac{1}{w(m+2)} \cdot mw = \frac{m}{m+2} \geq \frac{n}{n+2} > q$.

Part II: $\alpha\beta \subset 1^*$. We want to show that if $t \in \alpha\beta$ then $t < 1$. Without loss of generality assume $t > 0$ otherwise there is nothing to prove. Since $t \in \alpha\beta$ then $t \leq ab$ for some $a \in \alpha$ and $b \in \beta$ and $a, b > 0$. Now by definition of β, there exists $r > 1$ such that $\frac{1}{br} \notin \alpha$, i.e., $\frac{1}{br} > a$. This implies that $1 > \frac{1}{r} > ab \geq t$.

4. The rest of the problems

Chapter 1, Exercise 4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18.

5. The last problem

Take any real number $t \leq 0$, and let $p \leq t$ and $q \leq t$. Let $a \geq 0$. Show that

$$p\sqrt{a^2 + (y+b)^2} + q\sqrt{a^2 + (y-b)^2} - 2ty \leq -a\left(\sqrt{p^2 - t^2} + \sqrt{q^2 - t^2}\right)$$

holds true for any real number $b \in R$ and any $y \geq 0$.

Solution: Amir gave a nice solution. Here is a similar one. We will apply Cauchy–Schwarz inequality twice:

$$\sqrt{A^2 + B^2}\sqrt{B^2 + D^2} \geq |AB| + |CD|$$

for any real numbers $A, B, C, D \in \mathbb{R}$. Indeed we have

$$\begin{aligned}
&p\sqrt{a^2 + (y+b)^2} + q\sqrt{a^2 + (y-b)^2} - 2ty = \\
&-\sqrt{(p^2 - t^2) + t^2}\sqrt{a^2 + (y+b)^2} - \sqrt{(q^2 - t^2) + t^2}\sqrt{a^2 + (y-b)^2} - 2ty \leq \\
&-a\sqrt{p^2 - t^2} - |t(y+b)| - a\sqrt{q^2 - t^2} - |t(y-b)| - 2ty \leq -a\left(\sqrt{p^2 - t^2} + \sqrt{q^2 - t^2}\right).
\end{aligned}$$

In the last step we used the triangle inequality $-|t(y+b)| - |t(y-b)| - 2ty \leq 0$.