Name: ______________________ Signature: ______________________

There are no calculators or notes allowed. You will be given exactly 120 min. for this exam.
Please raise your hand if you have any questions and I will come to you. Show all your work to receive credit.

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Max	8	7	9	8	9	9	50

1. (4+4=8 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that for any rational $p \in \mathbb{Q}$ we have $f(p) > 0$.

   (a) Does it imply that $f(x) \geq 0$ for all $x \in \mathbb{R}$?

   (b) Does it imply that $f(x) > 0$ for all $x \in \mathbb{R}$?

   Solution: (a) Yes. Assume contrary that $f(s) < 0$ for some $s \in \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we can choose $\{q_j\}_{j\geq 1}$, sequence of rational numbers, such that $\lim_{j\to\infty} q_j = s$. By continuity $\lim_{j\to\infty} f(q_j) = f(s) < 0$. This means that for any $\varepsilon > 0$ there exists $N > 0$ such that for all $j \geq N$ we have $|f(s) - f(q_j)| < \varepsilon$ as soon as $j \geq N$. Now, $f(s)$ is negative number and $f(q_j)$ are positive. If we pick $\varepsilon = \frac{f(s)}{2}$ then we come to a contradiction.

   (b) No, take $f(x) = |x - \sqrt{2}|$.

2. (7 points)

   Does the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converge?

   Solution. Let us apply the ratio test

$$\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}.$$

   The ratio converges to $\frac{1}{e} < 1$. So the series must converge .

3. (9 points) Let K be a compact set in a metric space X, i.e.. from any open cover of K one can extract finite sub-cover. Show that K is closed. (hint: prove that the complement of K is open).

   Solution: Let us follows the hint. Pick $x \in K^c$. We should find an open neighborhood of x which does not intersect with K. For any $p \in K$, let $d(x,p) > 0$ be a distance between x and p. Let $N_{\frac{d(x,p)}{3}}(x)$ and $N_{\frac{d(x,p)}{3}}(p)$ be neighborhoods of radius $\frac{d(x,p)}{3}$ centered at points x and p correspondingly. Clearly they do not intersect with each other. Now $K \subset \cup_{p\in K} N_{\frac{d(x,p)}{3}}(p)$. We can extract finite sub-cover, say we find $p_1, p_2 \ldots, p_n \in K$ such that $K \subset N_{\frac{d(x,p_1)}{3}}(p_1) \cup \ldots \cup N_{\frac{d(x,p_n)}{3}}(p_n)$. Choose $r = \min\{\frac{d(x,p_1)}{3}, \ldots, \frac{d(x,p_n)}{3}\}$. Then $N_r(x) = \cap_{j=1}^n N_{\frac{d(x,p_j)}{3}}(x)$ is an open neighborhood of x which does not intersect with K.

4. (8 points)

Let $f : (0,1) \to \mathbb{R}$ be a uniformly continuous function. Show that f must be bounded, i.e., there exists $M > 0$ such that $|f(x)| < M$ for all $x \in (0,1)$.

Solution.

Uniform continuity says that for, say $\varepsilon = 1$, there exists $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$. Take N very large, and divide $(0,1)$ into N equal subintervals, say $(0, \frac{1}{N}], [\frac{1}{N}, \frac{2}{N}], \ldots, [\frac{N-1}{N}, 1)$. Each of them is of length $\frac{1}{N}$. So choose $N > 0$ so that $\frac{1}{N} < \delta$. Now pick any points $x_1 \in (0, \frac{1}{N}]$, $x_2 \in [\frac{1}{N}, \frac{2}{N}], \ldots, x_N \in [\frac{N-1}{N}, 1)$, and let $R = \max\{|f(x_1)|, \ldots, |f(x_N)|\}$. If $x \in (0,1)$, then there is a point x_j such that $|x - x_j| < \delta$ since x will fall into one of these N intervals. This implies that $|f(x) - f(x_j)| < 1$ which gives $|f(x)| < 1 + |f(x_j)| \le 1 + R$. Therefore $|f(x)| < 1 + R$ for all $x \in (0,1)$.

5. (9 points each).

   Let $f : (0,3) \to \mathbb{R}$ be a continuous function such that $f'(x), f''(x), ..., f^{(2019)}(x)$ exist and are continuous for all $x \in (0,3)$. Assume $f(1) = f'(1) = f''(1) = \ldots = f^{(2018)}(1) = 0$. Also suppose that $f(2) = 0$. Show that there exists a point $t \in (1,2)$ such that $f^{(2019)}(t) = 0$.

   Solution. Indeed, since $f(1) = f(2) = 0$, there exists a point $t_1 \in (1,2)$ such that $f'(t_1) = 0$ by mean value theorem. Next, $f'(1) = f'(t_1) = 0$, so there exists a point $t_2 \in (1, t_1)$ such that $f''(t_2) = 0$. In this way we construct points $t_1, t_2, \ldots, t_{2019}$. Notice that $f^{(2019)}(t_{2019}) = 0$, and $t_{2019} \in (1,2)$.

6. (9 points)

   Let $a_1, \ldots, a_{10}$ be nonnegative numbers such that $\sum_{k=1}^{10} a_k = 1$. Also assume that $\sum_{k=1}^{10} \frac{a_k}{k} > \frac{3}{5}$. Show that $\sum_{k=5}^{10} a_k < \frac{1}{2}$.

   Solution.

   $1 = \sum_{k=1}^{10} a_k = \sum_{k=1}^{4} a_k + \sum_{k=5}^{10} a_k = A + B$. We have $A + B = 1$. Then

$$\frac{3}{5} < \sum_{k=1}^{10} \frac{a_k}{k} = \sum_{k=1}^{4} \frac{a_k}{k} + \sum_{k=5}^{10} \frac{a_k}{k} \leq A + \frac{B}{5} = 1 - B + \frac{B}{5} = 1 - \frac{4B}{5}$$

   The latter implies $\frac{4B}{5} < \frac{2}{5}$, and therefore $B < \frac{1}{2}$.