Name: ____________________ Signature: ____________________

There are no calculators or notes allowed. You will be given exactly 120 min. for this exam.
Please raise your hand if you have any questions and I will come to you. Show all your work to receive credit.

1. (10 points) Calculate the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}$$

Solution: We have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+(k/n)} = \int_0^1 \frac{1}{1+x} dx = \ln(2)$$

2. (10 points)

Let f be a real valued function on $\mathbb{R}$ which is infinitely many times differentiable. Assume that $|f'''| \leq 1$ on $\mathbb{R}$. Show that there exist constants $A, B, C, D \geq 0$ such that

$$|f(x)| \leq A + B|x| + Cx^2 + D|x^3| \quad \text{for all} \quad x \in \mathbb{R}$$

Solution: By Taylor's formula

$$|f(x)| = \left| f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(t)}{3!}x^3 \right| \leq |f(0)| + |f'(0)||x| + \left| \frac{f''(0)}{2!} \right| x^2 + \frac{1}{3!}|x^3|$$

3. (5+5 points)

(a) Give an example of a sequence $S_{n,m}$ such that both $\lim_{m\to\infty}(\lim_{n\to\infty})S_{n,m}$ and $\lim_{n\to\infty}(\lim_{m\to\infty})S_{n,m}$ exist but they are not equal.

(b) What does it mean that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $(a,b) \subset \mathbb{R}$?

Solution:

(a) take $S_{n,m} = \frac{m}{n+m}$.

(b) it means that $s_N(x) = \sum_{n=1}^{N} f_n(x)$ converge uniformly to some function $s(x)$ on (a,b), i.e., there exists a function $s(x)$ on (a,b) such that for any $\varepsilon > 0$ there exists $N > 0$ such that $\sup_{x\in(a,b)} |s_k(x) - s(x)| < \varepsilon$ for all $k \geq N$.

4. (10 points)

Show that the series

$$\sum_{k=1}^{\infty} \frac{1}{1 + x^{2k+1}}$$

converges uniformly on $[2, 3]$.

Solution: Notice that $\frac{1}{1+x^{2k+1}} \leq \frac{1}{1+2^{2k+1}} \leq \frac{1}{2^{2k+1}}$ on $[2, 3]$. Therefore the claim follows by using Weierstrass test and the fact that $\sum_{k\geq 1} \frac{1}{2^{2k+1}}$ converges (geometric series).

5. (10 points each). Suppose we are given a family of differentiable functions $\{f_n\}_{n\geq 1}$ on $[0,1]$ such that $|f_n'(x)| \leq 10$ for all $x \in [0,1]$ and all $n \geq 1$. Assume that $|f_n(x)| \leq e^x$ for all $x \in [0,1]$ and all $n \geq 1$. Does $\{f_n\}_{n\geq 1}$ contain a uniformly convergent subsequence?

   Remark: if you want to use a certain theorem then please clearly formulate the theorem with all its assumptions (you do not have to prove the theorem).

   Solution: The family $\{f_n\}$ is equicontinuous. Indeed $|f_n(x) - f_n(y)| = |f_n'(c)||x-y| \leq 10|x-y|$. Also $\{f_n\}_{n\geq 1}$ is pointwise bounded because $|f_n(x)| \leq e^x$. Therefore $\{f_n\}$ contains uniformly convergent subsequence by Arzelá–Ascoli theorem.

6. (10 points)

Show that there exists sequence of polynomials $\{P_n\}_{n\geq 1}$ such that P_n uniformly converges to $f(x) = e^{\sin(x)} - 1$ on $[-1, 1]$. Moreover $P_n(0) = 0$ for all $n \geq 1$.

Solution: By Stone–Weierstrass there exists a sequence of polynomials $Q_n(x)$ such that $Q_n(x)$ converges uniformly to $f(x)$. Since $f(0) = 0$ then $Q_n(0) \to 0$ as $n \to \infty$. Consider polynomials $P_n(x) = Q_n(x) - Q_n(0)$. Then

$$|f - P_n| = |f(x) - Q_n(x)| + |Q_n(0)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

as soon as $n \geq N$ for sufficiently large N.

7. (10 points)

The Gamma functions $\Gamma(x)$ is defined as $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. Show that Γ is log-convex on $(0, \infty)$.

Soluiton: let us apply Hölder inequality

$$\Gamma(x)^\alpha \Gamma(y)^{1-\alpha} = \left(\int_0^\infty t^{x-1}e^{-t}dt\right)^\alpha \left(\int_0^\infty t^{y-1}e^{-t}dt\right)^{1-\alpha} \geq \int_0^\infty t^{\alpha x+(1-\alpha)y-1}e^{-t}dt = \Gamma(\alpha x + (1-\alpha)y)$$

for all $\alpha \in (0,1)$ (and in fact $\alpha \in [0,1]$).

8. (10 points) Find the asymptotic behaviour of the the sum

$$\sum_{k=0}^{n} \frac{(-1)^k}{n+k+1}\binom{n}{k}$$

when n goes to infinity.

Solution: let us apply Laplace method.

$$\sum_{k=0}^{n} \frac{(-1)^k}{n+k+1}\binom{n}{k} = \sum_{k=0}^{n} \int_0^1 (-1)^k \binom{n}{k} t^{n+k} dt = \int_0^1 (t(1-t))^n dt = \int_0^1 e^{n\ln(t(1-t))} dt$$

Since $g(t) := \ln(t(1-t))$ attains unique global maximum at point $t = 1/2$ (and $g'(1/2) = 0$) and outside of a neighborhood of $t = 1/2$ we have $g(1/2) - \delta > g(x)$ for some $\delta > 0$ and all $x \in (N_\varepsilon(1/2))^c$ we can apply Laplace method. We have $g(1/2) = \ln(1/4)$, and $g''(t) = -\frac{1}{t^2} - \frac{1}{(1-t)^2}$, so $g''(1/2) = -8$ we have

$$\int_0^1 e^{n\ln(t(1-t))} dt \sim e^{ng(1/2)} \sqrt{\frac{2\pi}{n|g''(1/2)|}} = e^{n\ln(1/4)} \sqrt{\frac{2\pi}{8n}} = \frac{\sqrt{\pi}}{2 \cdot 4^n \sqrt{n}} = \frac{\sqrt{\pi}}{2^{2n+1}\sqrt{n}}$$

as n goes to infinity.