MIDTERM 205B, WINTER 2019

Problem 1. *Give an example of a bounded function on* $[0, 1]$ *which is not Riemann integrable.*

Problem 2. *Give an example of sequence of continuous functions* $\{f_n\}_{n\geq 1}$ *and* $\{g_n\}_{n\geq 1}$ *defined on* $(0,1)$ *such that* f_n *converges uniformly to* f *on* $(0,1)$*, also* g_n *converges uniformly to* g *on* $(0,1)$ *but* $f_n g_n$ *does not converge uniformly to* fg *on* $(0,1)$*.*

Problem 3. *Let f be a continuous function $[0,1]$. Show that if $\int_0^1 f^2(x)dx = 0$ then $f = 0$ on $[0,1]$.*

Problem 4. *Let f be a continuous function on $[0,1]$ such that $\int_0^1 f(x)x^n dx = 0$ for all $n \geq 0$. Show that $f = 0$ on $[0,1]$.*

Hint: use Stone–Weierstrass theorem and the previous problem.

Problem 5. *Take any two positive continuous functions* f, g *on* $[0,1]$ *such that the graphs of these functions cross at exactly one point* $s \in (0,1)$*. Assume* $f(0) < g(0)$ *and* $f(1) > g(1)$*, and* $\int_0^1 f = \int_0^1 g$*. Show that* $\int_0^1 x f(x)dx \geq \int_0^1 x g(x)dx$*.*

Hint: Consider $f(x) - g(x)$*. Try to multiply the latter expression by some factor to make the result nonnegative and finally integrate it over the interval* $[0,1]$*.*