PROBLEM 5

Let G be an open set in $\mathbb{R}$ which is not upper bounded. Does there exist $b > 0$ such that $\{nb\}_{n\geq 1} \cap G$ is infinite?

Proof. The answer is yes.

Pick any $0 < p < q$ and $n_0 > 0$. Then $\cup_{n\geq n_0}(np, nq)$ contains (C, ∞) for some $C > 0$ (why?). Indeed, notice that when n increases then the length of the segment (np, nq) is $n(q-p)$ which also increases to infinity. Let us look at two consecutive segments (np, nq) and $((n+1)p, (n+1)q)$. The distance between np and $(n+1)p$ is p which is fixed, therefore it is clear that starting from some sufficiently large $n_0 > 0$ we will have $(n+1)p \in (np, nq)$ for all $n \geq n_0$.

Since $(C, \infty) \cup G$ is nonempty it follows that there exists an integer $N_1 \geq n_0$ such that $(N_1 p, N_1 q) \cap G$ is not empty. This implies that there exists $x \in (p, q)$ such that $N_1 x \in G$. Since G is open there exist $0 < p_1 < x < q_1$ (in a tiny neighborhood of x) with $[p_1, q_1] \subset (p, q)$ such that $[N_1 p_1, N_1 q_1] \subset G$.

Indeed, $N_1 x \in G \cap (N_1 p, N_1 q)$. Since $G \cap (N_1 p, N_1 q)$ is open we have $(N_1 x - \delta, N_q x + \delta) \subset G \cap (N_1 p, N_1 q)$ for some small $\delta > 0$. Then $(x - \frac{\delta}{N}, x + \frac{N}{\delta}) \subset (p, q)$. Choose $[p_1, q_1] = [x - \frac{\delta}{2N}, x + \frac{\delta}{2N}]$

Next, consider $0 < p_1 < q_1$, and $n_1 > 0$ (with $n_1 > N_1$), and repeat the previous step. Thus we obtain the sequence of intervals $[p_1, q_1] \supset [p_2, q_2] \supset ....$ and sequence of strictly increasing integers $N_1 < N_2 < ...$ such that $[N_1 p_1, N_1 q_1], [N_2 p_2, N_2 q_2], \ldots, \subset G$. Notice that for $b \in \cap_{k\geq 1}[p_k, q_k]$ we have $N_1 b, N_2 b, \ldots \in G$.

□