Sharp estimates in harmonic analysis

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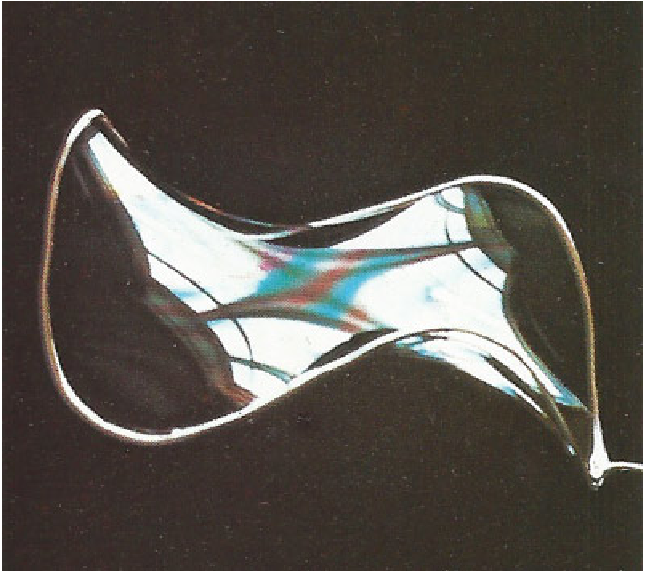
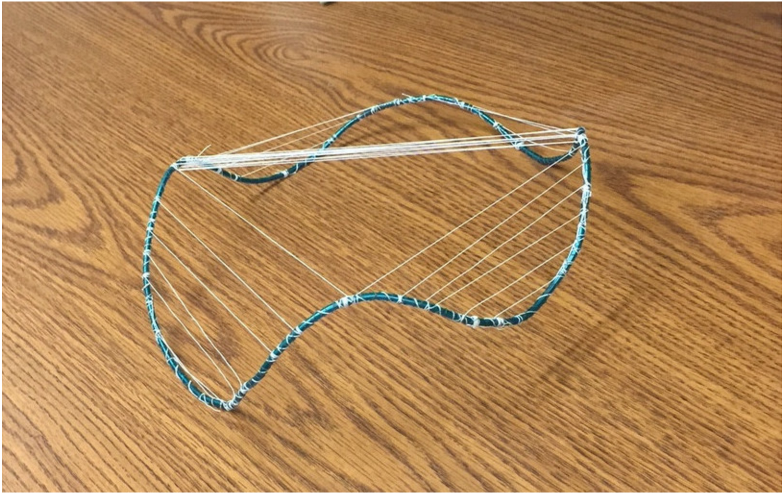
Bellman function for extremal problems in BMO.
(joint with N. Osipov, D. Stolyarov, P. Zatitskiy, V. Vasyunin)

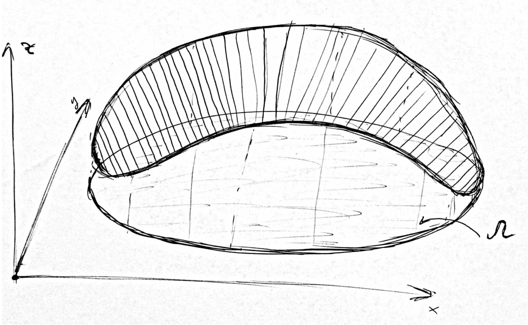
Surface of minimal area $k_1 + k_2 = 0$.



Minimal concave function $k_1 k_2 = 0$



Find the minimal concave function

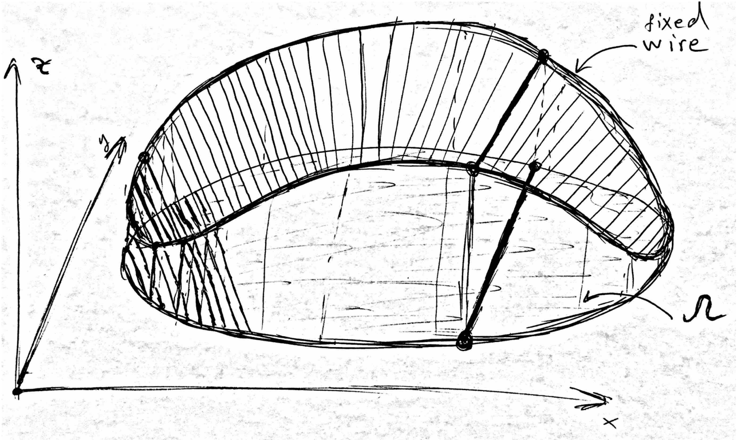


$B(x, y)$ is minimal concave function on Ω with $B|_{\partial\Omega} = f$.

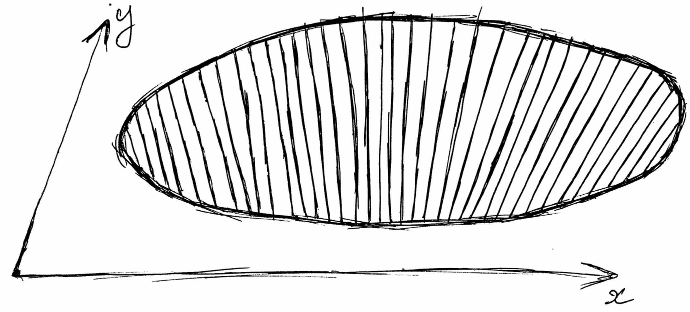
Problem

Given a wire find B.

Projection

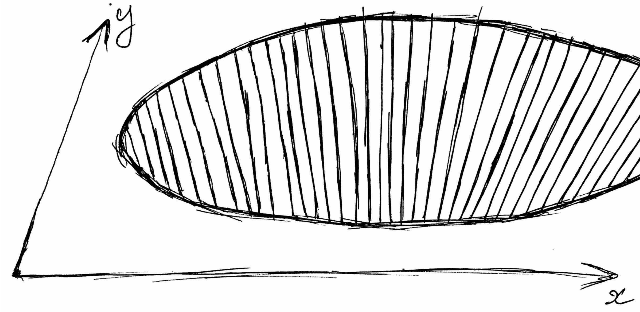


Foliation



This we call foliation.

Foliation

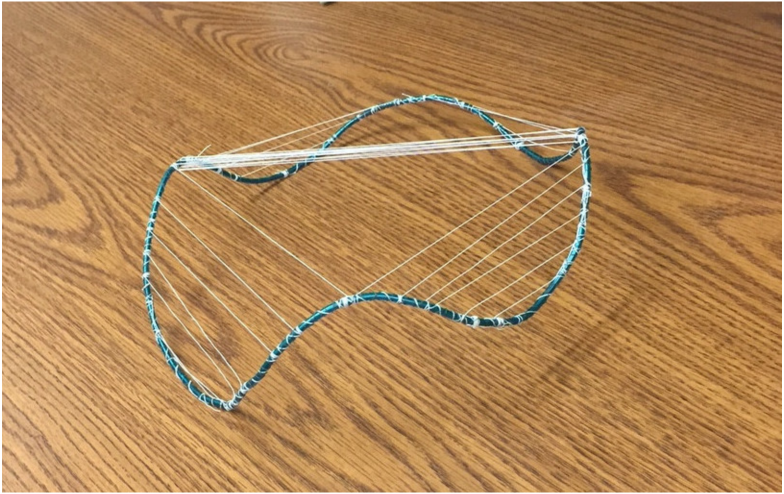


This we call foliation.

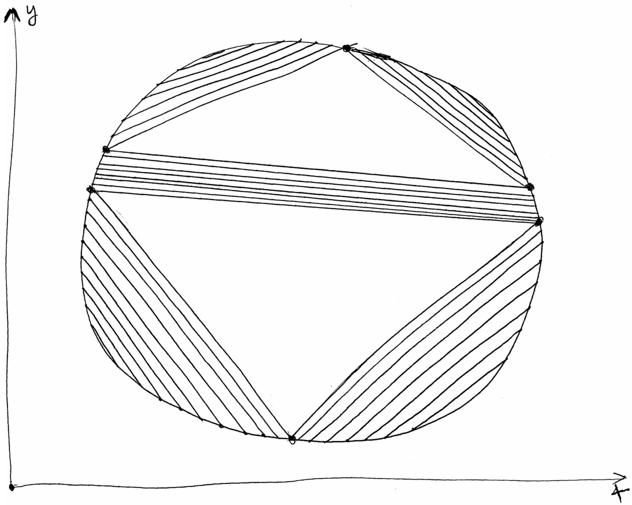
Problem (equivalent)

Given a wire find foliation.

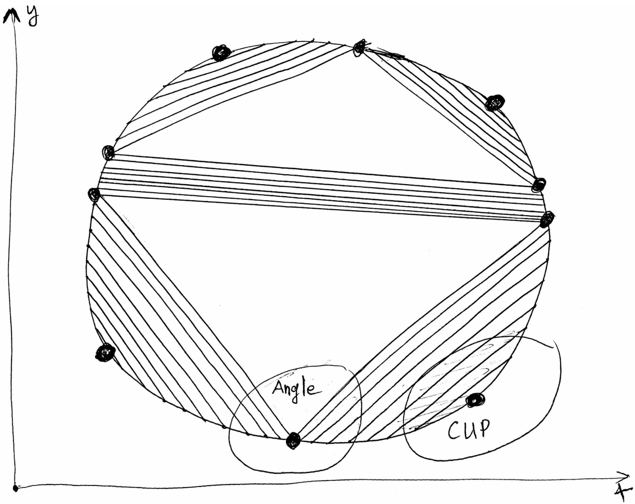
Concave envelope



Foliation



Angle and Cup



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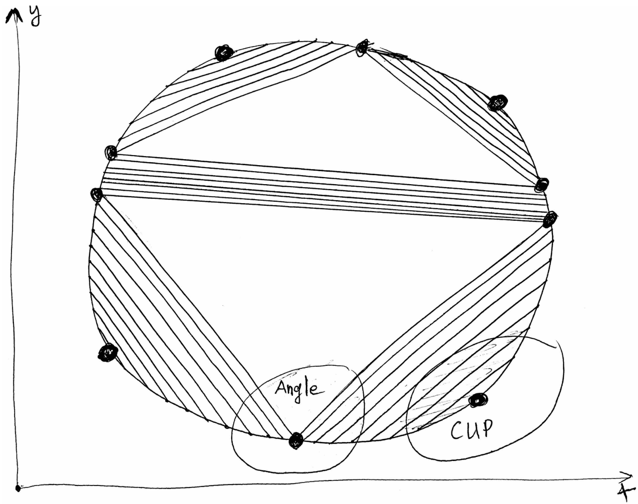
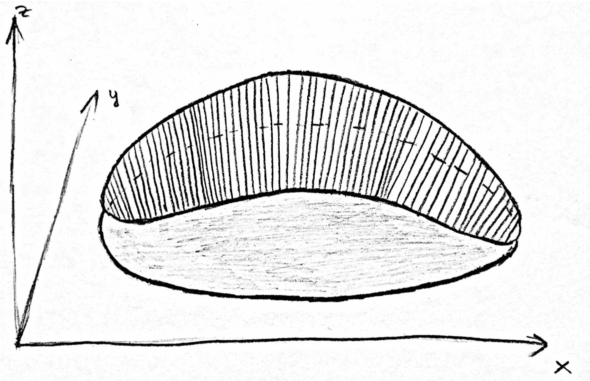


Demonstrate

Boundary value problem: homogeneous Monge–Ampère



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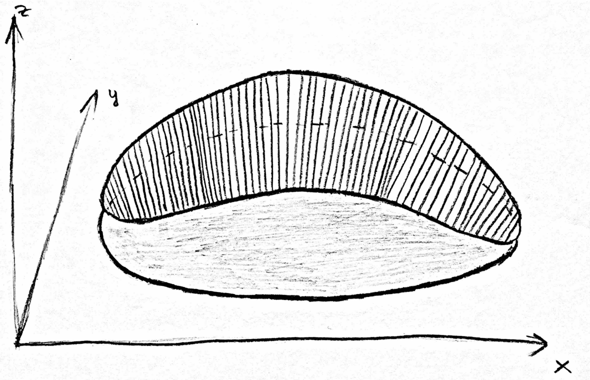
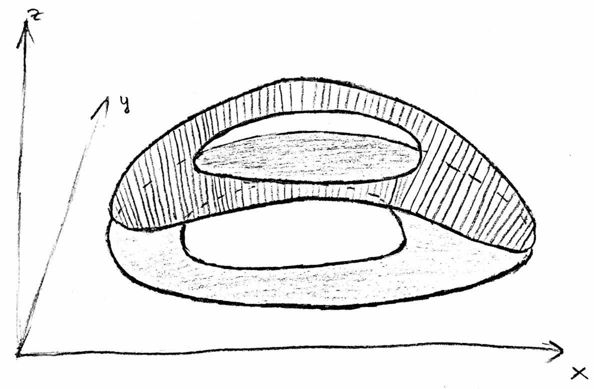
Boundary value problem: homogeneous Monge–Ampère



$B(x, y)$ is minimal concave function on Ω with $B|_{\partial\Omega} = f$.
What if $B(x, y)$ is minimal concave function on $\Omega \setminus D$?

Erasing a hole



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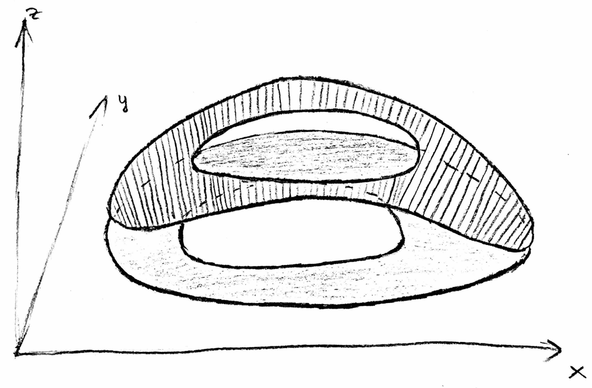
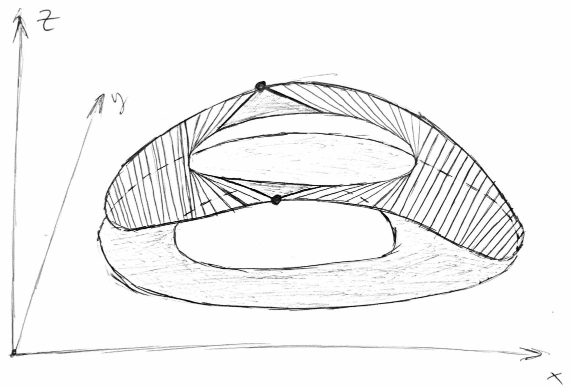
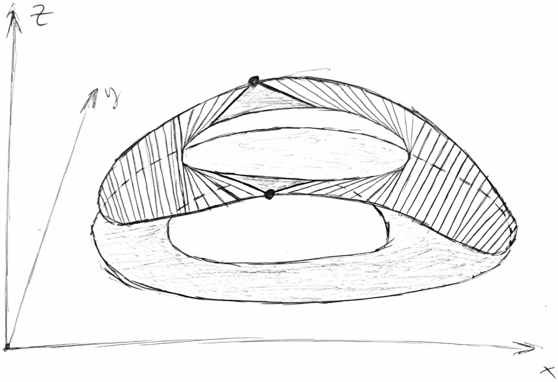


Will not be minimal!

Erasing a hole: $\Omega \setminus D$

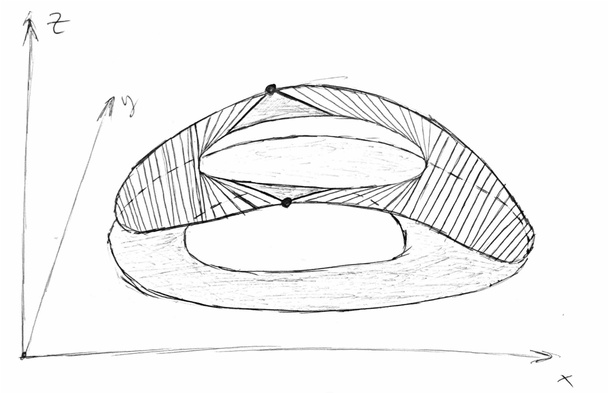


Erasing a hole: $\Omega \setminus D$



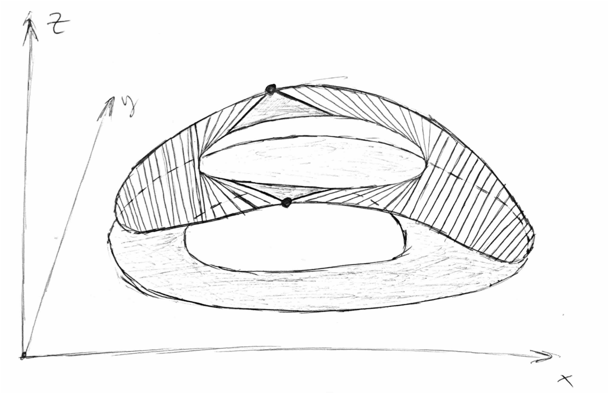
$\mathrm{Hess}\, B \leq 0$ on $\Omega \setminus D$

Erasing a hole: $\Omega \setminus D$



$\mathrm{Hess}\, B \leq 0$ on $\Omega \setminus D$
$B|_{\partial\Omega} = f$.

Erasing a hole: $\Omega \setminus D$



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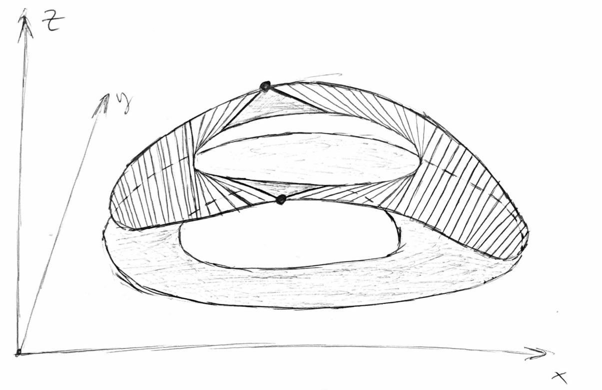
$B|_{\partial\Omega} = f$.
B is minimal among all such solutions.



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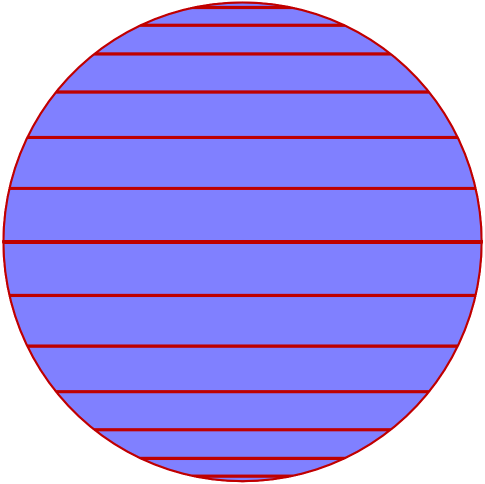
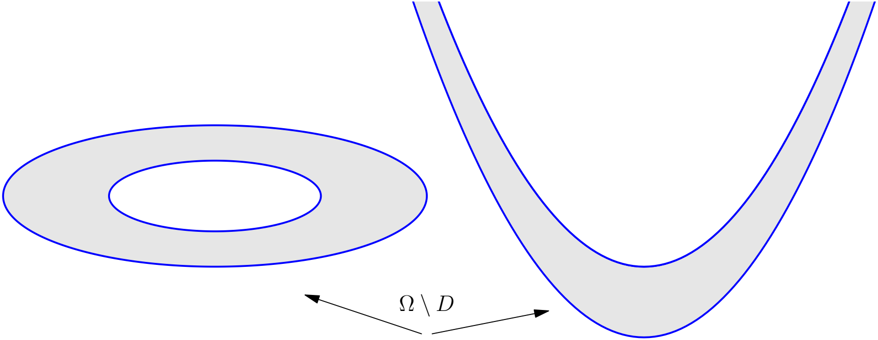
$\det(\mathrm{Hess}\ B) = 0$ on $\Omega \setminus D$;

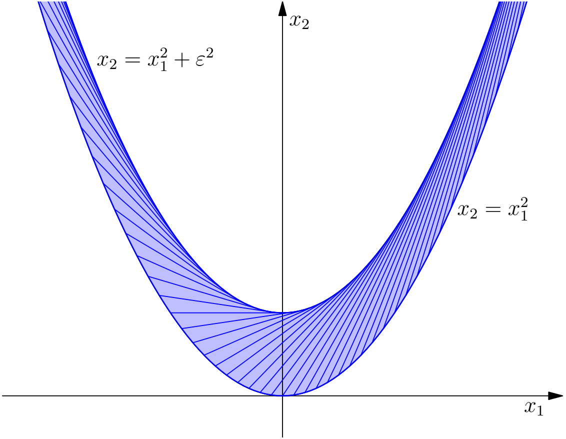
Evolution of the surface



Annulus and parabolic strips

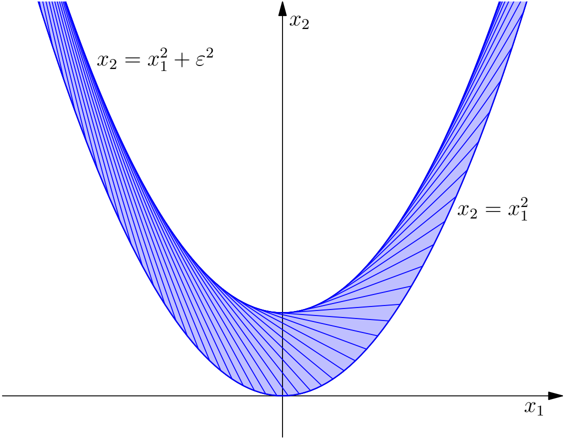


Left tangent lines 2D



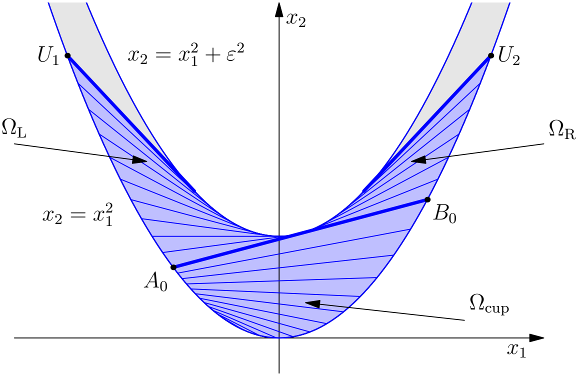
Left tangent lines 3D

Right tangent lines 2D



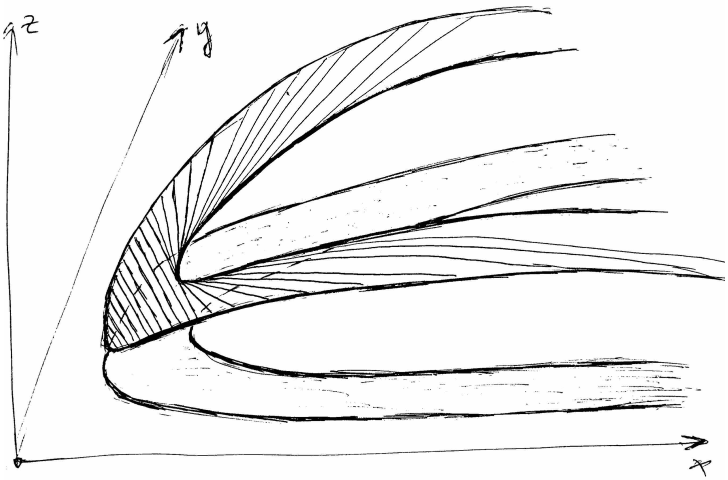
Right tangent lines 3D

Cup

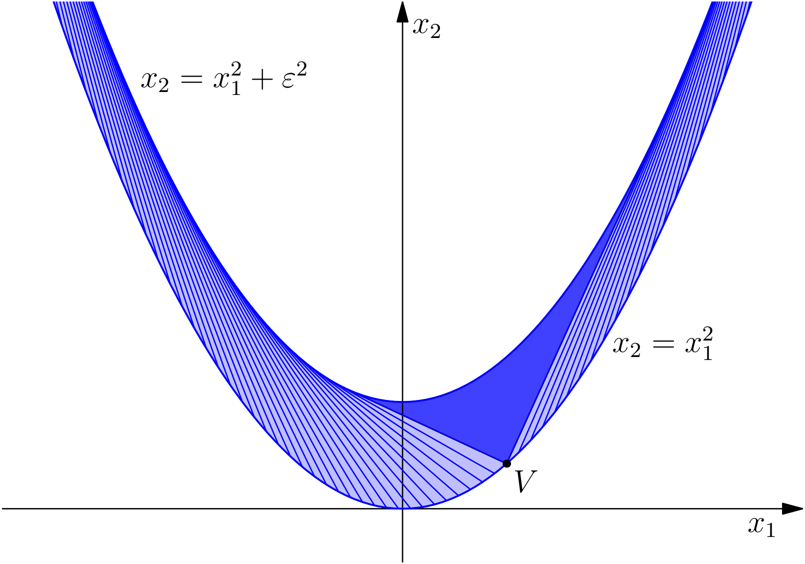


x_2
U_1
$x_2 = x_1^2 + \varepsilon^2$
U_2
Ω_{L}
Ω_{R}
$x_2 = x_1^2$
B_0
A_0
Ω_{cup}
x_1

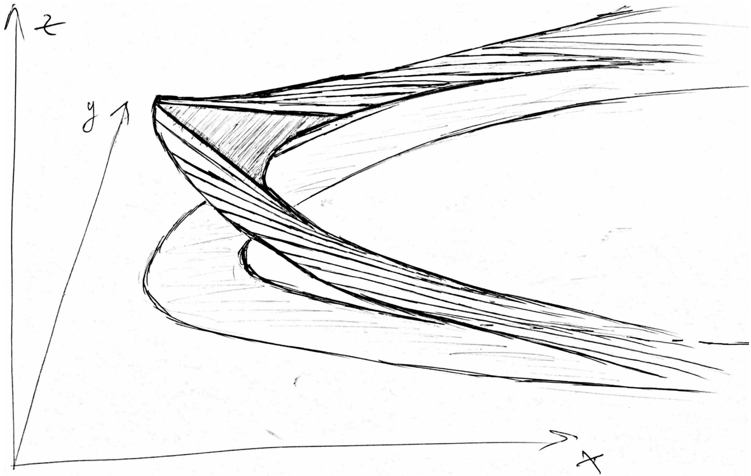
Cup



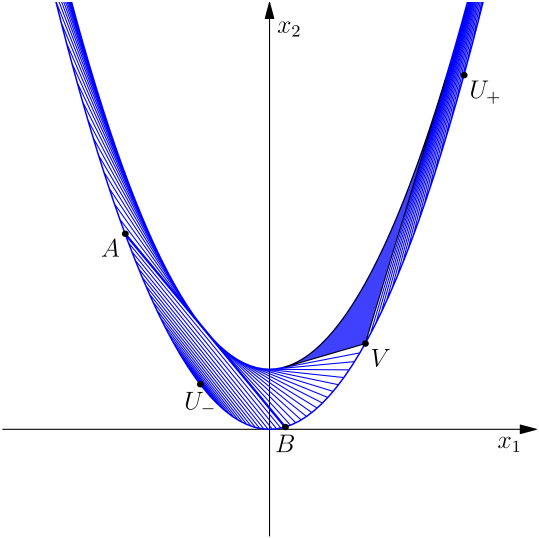
Angle



Angle



Angle and cup



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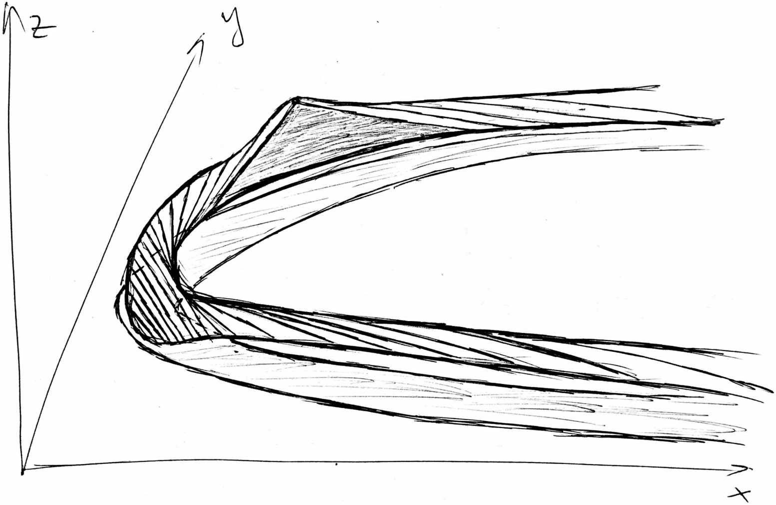
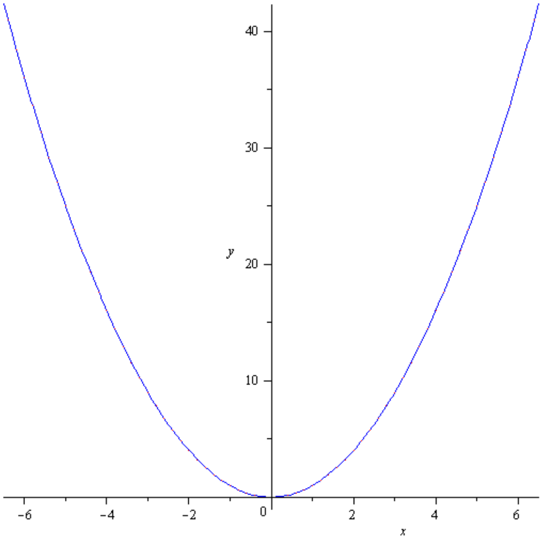
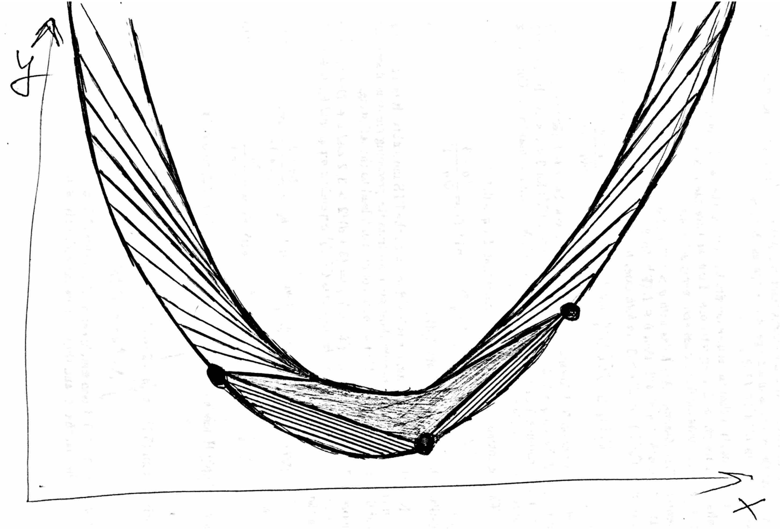


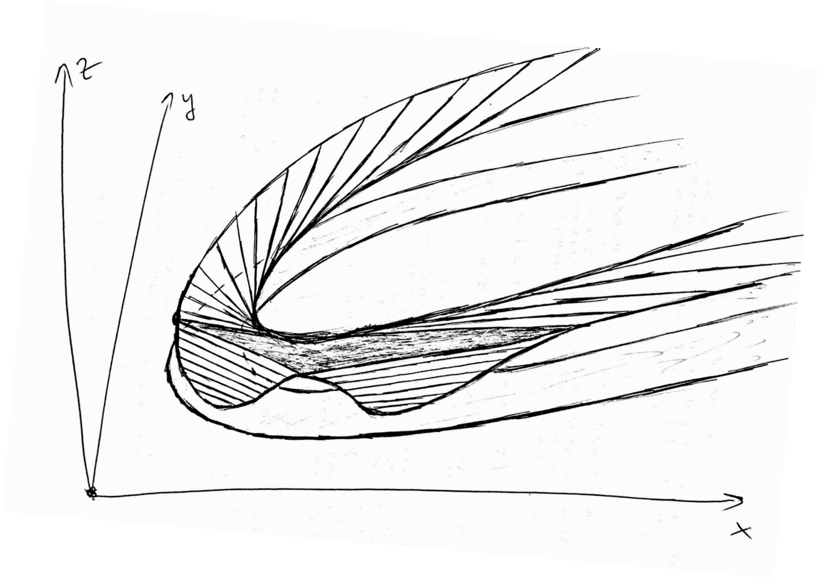
Angle and cup: evolution



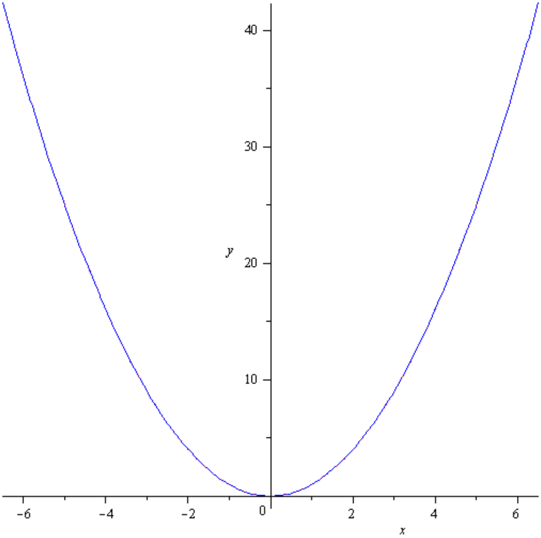
Two cups



Two cups



Two cups: evolution



John–Nirenberg inequality and BMO space

$$\sup_{I \subseteq [0,1]} \frac{1}{|I|} \left| \left\{ t \in I \ : \ \left| \varphi(t) - \frac{1}{|I|} \int_I \varphi \right| \geq \lambda \right\} \right| \leq c_1 e^{-c_2 \lambda / \|\varphi\|_{\mathrm{BMO}^2([0,1])}}$$

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Question

Find sharp constants $c_1, c_2 > 0$.

$$B_{\varepsilon}(x_1,x_2) \stackrel{\text{def}}{=} \sup_{\varphi\,:\,\|\varphi\|_{BMO}\leq\varepsilon} \left\{ \int_0^1 f(\varphi(t))dt \;:\; \int_0^1 \varphi = x_1,\ \int_0^1 \varphi^2 = x_2, \right\}$$

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Example: $f(t) = \mathbb{1}_{(-\infty,-\lambda)\cup(\lambda,\infty)}(t)$ - John–Nirenberg inequality.

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Main results (IOSVZ):

1. B_ε is minimal concave function on $\Omega_0 \setminus \Omega_\varepsilon$ with $B|_{\partial\Omega_0} = f$ where $\Omega_\varepsilon = \{(x_1,x_2) \subset \mathbb{R}^2 \;:\; x_2 \geq x_1^2 + \varepsilon^2\}$
2. For each ε we find B_ε (stationary algorithm).
3. We described evolution of $B_\varepsilon, \varepsilon \geq 0$ (dynamical approach).

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The torsion of $(t, t^2, f(t))$ changes sign finitely many times.

Other applications

Theorem (Ivanisvili–Jaye–Nazarov)

For any $p > 1$ and $n \geq 1$ there exists $A(p,n) > 1$ such that

$$\int_{\mathbb{R}^n} \left(\sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy \right)^p dx \geq A(p,n) \int_{\mathbb{R}^n} |f(x)|^p dx, \quad \forall f \in L^p$$

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Theorem (P.I.)

$$\sup_{\varepsilon_I \in \{-1,1\}} \int_0^1 \left(\left(\sum_{I \in D} \varepsilon_I \langle f, h_I \rangle h_I \right)^2 + \tau^2 f^2 \right)^{p/2} dx \leq C \int_0^1 |f|^p dx$$

Settles an open problem of Boros–Janakiraman–Volberg

The end

Thank you