Burkholder's martingale transform

Paata Ivanishvili

Michigan State University

Abstract

We find the sharp constant $C = C(\tau, p, |\mathbb{E}G/\mathbb{E}F|)$ of the following inequality $\|(G^2+\tau^2F^2)^{1/2}\|_p \le C\|G\|_p$, where G is the transform of a martingale F under a predictable sequence ε with absolute value 1, $1 < p < 2$, and τ is any real number.

Martingale transform

We consider the probability space $([0,1], \mathcal{B}, dx)$. Let $\mathcal{M}_n$ be the σ-algebra generated by the dyadic intervals

$$I_{n,j} \overset{\text{def}}{=} \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right), \quad 0 \le j \le 2^n - 1, \quad n \ge 1$$

Set $\mathcal{M}_0 = [0,1]$, and $F_n \overset{\text{def}}{=} \mathbb{E}(F|\mathcal{M}_n)$ for $F \in L^1([0,1])$. We say that $\{F_n\}_{n=0}^{\infty}$ is a dyadic martingale constructed by F.

DEFINITION. Let F and G be real valued integrable functions. If the dyadic martingale $\{G_n\}$ constructed by G satisfies $|G_{n+1}-G_n| = |F_{n+1}-F_n|$ for each $n \ge 0$, then G is called martingale transform of F.

In [1] Burkholder proved that for if $|G_0| \le |F_0|$, for any p, $1 < p < \infty$, we have

$$\|G_n\|_{L^p} \le (p^*-1)\|F_n\|_{L^p} \quad \forall n \ge 0, \tag{1}$$

where $p^* - 1 = \max\{p-1, \frac{1}{p-1}\}$, and p^*-1 in (1) is sharp.

In [2] a bit more general estimate was obtained by Bellman function technique and Monge–Ampère equation, namely, estimate (1) holds if and only if

$$|G_0| \le (p^*-1)|F_0|. \tag{2}$$

In [3], a perturbation of the martingale transform was investigated. Namely, under the same assumptions as (2) it was proved that for $2 \le p < \infty$, $\tau \in \mathbb{R}$, we have

$$\|(G_n^2+\tau^2F_n^2)^{1/2}\|_{L^p} \le ((p^*-1)^2+\tau^2)^{1/2}\|F_n\|_{L^p}, \quad \forall n \ge 0, \tag{3}$$

where the constant $((p^*-1)^2+\tau^2)^{1/2}$ is sharp. It was also announced as proven (however proof is wrong) that the same sharp estimate holds for $1 < p < 2$, $|\tau| \le 0.5$ and the case $1 < p < 2$, $|\tau| > 0.5$ was left open.

Our main results

Further we assume that $1 < p < 2$ and $\tau \in \mathbb{R}$. We find the following Bellman function

$$H(x_1,x_2,x_3) = \sup_{F,G}\{\mathbb{E}(G^2+\tau^2F^2)^{p/2} : \mathbb{E}F = x_1, \mathbb{E}G = x_2, \mathbb{E}|F|^p = x_3, |F_{n+1}-F_n| = |G_{n+1}-G_n|, n \ge 0\}.$$

As a corollary we obtain the following theorem. Set

$$u(z) \overset{\text{def}}{=} \tau^p(p-1)\left(\tau^2+z^2\right)^{(2-p)/2} - \tau^2(p-1) + (1+z)^{2-p} - z(2-p) - 1.$$

Theorem. *Let $1 < p < 2$, and let $\{G_n\}_{n=0}^{\infty}$ be a martingale transform of $\{F_n\}_{n=0}^{\infty}$. Set $\beta' = \frac{|G_0|-|F_0|}{|G_0|+|F_0|}$. The following estimates are sharp*

1. If $u\left(\frac{1}{p-1}\right) \le 0$ then

$$\|(\tau^2F_n^2+G_n^2)^{1/2}\|_{L^p} \le \left(\tau^2 + \max\left\{\left|\frac{G_0}{F_0}\right|, \frac{1}{p-1}\right\}^2\right)^{\frac{1}{2}}\|F_n\|_{L^p}, \quad \forall n \ge 0.$$

2. If $u\left(\frac{1}{p-1}\right) > 0$ then

$$\|(\tau^2F_n^2+G_n^2)^{1/2}\|_{L^p}^p \le C(\beta')\|F_n\|_{L^p}^p, \quad \forall n \ge 0,$$

where $C(\beta')$[1] is continuous nondecreasing, and it is defined by the following way:

$$C(\beta') \overset{\text{def}}{=} \begin{cases} \left(\tau^2 + \left[\frac{1+\beta'}{1-\beta'}\right]^2\right)^{p/2}, & \beta' \ge s_0; \\ \dfrac{\tau^p}{1 - \frac{2^{2-p}(1-s_0)^{p-1}}{(\tau^2+1)(p-1)(1-s_0)+2(2-p)}}, & \beta' \le -1+\frac{2}{p}; \\ C(\beta') & \beta' \in (-1+2/p, s_0); \end{cases}$$

where $s_0 \in (-1+2/p, 1)$ is the solution of the equation $u\left(\frac{1+s_0}{1-s_0}\right) = 0$.

Proof. Here is the sketch.

Proposition. *The function H satisfies the following properties.*

1. H is defined in the domain $\Omega \overset{\text{def}}{=} \{(x_1,x_2,x_3) \in \mathbb{R}^3 : |x_1|^p \le x_3\}$.

2. H is diagonally concave function i.e. it is concave function in the domains $T_{\pm}(A) = \Omega \cap \{(x_1,x_2,x_3) : x_1 \pm x_2 = A\}$ for all $A \in \mathbb{R}$.

3. H has the following boundary condition $H(x_1,x_2,|x_1|^p) = (x_2^2+\tau^2x_1^2)^{p/2}$.

4. If a continuous function U satisfies the properties 1,2,3 then $U \ge H$.

It turns out that H should be a minimal diagonally concave function in Ω with the given boundary data. The function H is "homogeneous" in the following sense $H(\lambda x_1, \lambda x_2, \lambda^p x_3) = \lambda^p H(x_1,x_2,x_3)$. So one can "reduce" the question of finding the function H to the finding of a minimal concave function of two variables with the given Neumann and Dirichlet boundary conditions.

Construction of the Bellman function H.

Firstly we develop theory of minimal concave functions of two variables with the given boundary condition (see [5], [4]). We see that the torsion of the boundary curve plays crucial role. We construct *candidate* $M(y_1,y_2)$ for a minimal concave function in the domain

$$\Omega_1 \overset{\text{def}}{=} \{(y_1,y_2) : -1 \le y_1 \le 1, \ (1-y_1)^p \le y_2\}$$

with the following boundary conditions:

$$M(y_1,(1-y_1)^p) = ((1+y_1)^2 + \tau^2(1-y_1)^2)^{p/2}, \quad y_1 \in [-1,1],$$
$$0 = pM(-1,y_2) + 2\frac{\partial M}{\partial y_1}(-1,y_2) - py_2\frac{\partial M}{\partial y_2}(-1,y_2), \quad \text{for} \quad y_2 \ge 0,$$
$$0 = pM(1,y_2) - 2\frac{\partial M}{\partial y_1}(1,y_2) - py_2\frac{\partial M}{\partial y_2}(1,y_2), \quad \text{for} \quad y_2 \ge 0.$$

We find such function M among solutions of homogeneous Monge–Ampére equation.

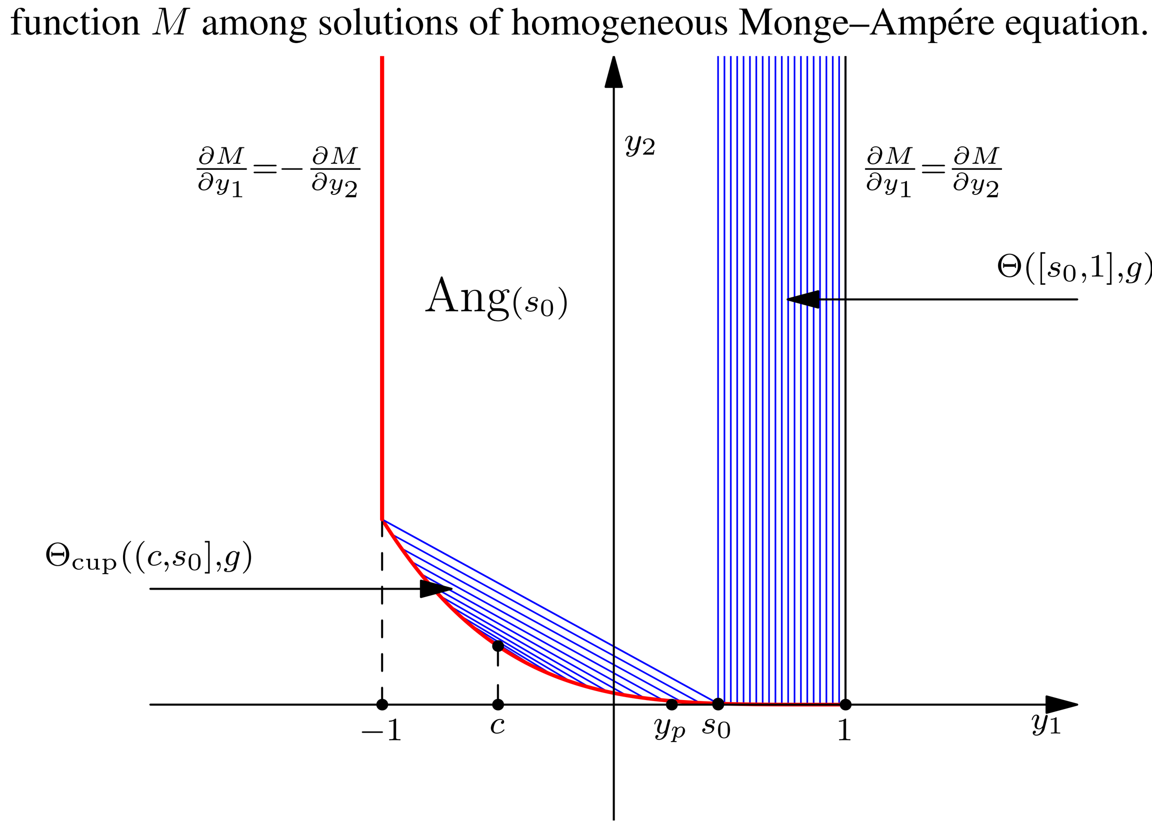

Figure 1: The function $M(y_1,y_2)$ is linear along line segments shown on the picture. The family of these line segments is called *foliation*. The function M is linear in the domain $\text{Ang}(s_0)$. For a different parameters τ and p foliation is different. In this case we have $u\left(\frac{1}{p-1}\right) > 0$.

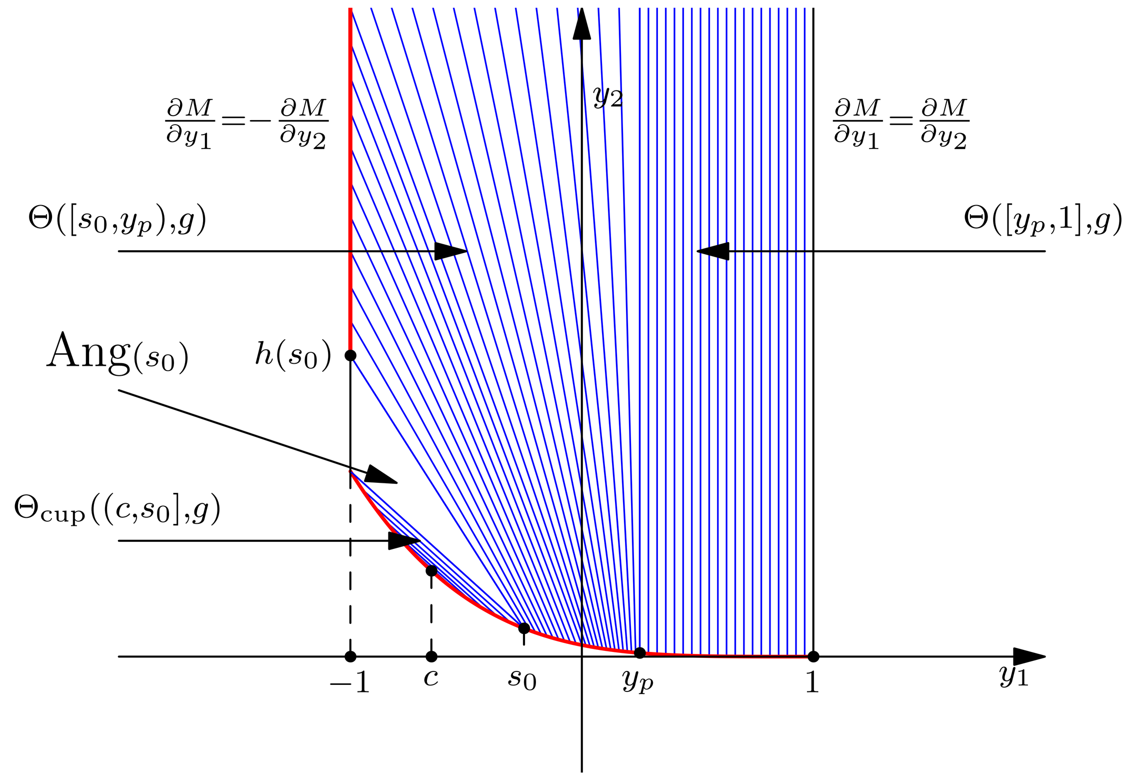

Figure 2: The given foliation corresponds to the case when $u\left(\frac{1}{p-1}\right) \le 0$.

Then we consider new function N defined in the domain $\Omega_2 \overset{\text{def}}{=} \{(y_1,y_2,y_3) \colon y_3 \ge 0, \ |y_1-y_2|^p \le y_3\}$ as follows

$$N(y_1,y_2,y_3) = y_1^p M\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right), \quad y_1 \ge 0.$$
$$N(y_1,y_2,y_3) = N(y_2,y_1,y_3) = N(-y_1,-y_2,y_3).$$

Finally we set *candidate* B for H as the following function

$$B(x_1,x_2,x_3) = N\left(\frac{x_1+x_2}{2}, \frac{x_2-x_1}{2}, x_3\right), \quad (x_1,x_2,x_3) \in \Omega.$$

After some technical computations it turns out that $B \in C^1(\Omega)$ and B is a diagonally concave function with the same boundary data as H, therefore by the proposition $B \ge H$.

Optimizers

In the end, by knowing foliation we construct the optimizers. Namely, given a point $\mathbf{x} = (x_1,x_2,x_3) \in \Omega$ and any $\varepsilon > 0$ we construct such a pair of functions (F,G) such that G is a martingale transform of F, $(\mathbb{E}F, \mathbb{E}G, \mathbb{E}|F|^p) = (x_1,x_2,x_3)$ and $\mathbb{E}(G^2+\tau^2F^2)^{p/2} \ge B(\mathbf{x}) - \varepsilon$, therefore we get $H \ge B$, thus $H = B$.

□

Conclusions and Questions

- Let G be a martingale transform of F, then for the given parameters τ, p, F_0, G_0 what is the sharp C such that $\|G + \tau F\|_p \le C\|G\|_p$?

References

[1] D. L. Burkholder, *Boundary value problem and sharp inequalities for martingale transforms,* Ann. Probab. 12 (1984), 647-702.

[2] V. Vasyunin, A. Volberg, *Burkholder's function via Monge-Ampère equation* Illinois Journal of Mathematics 54 (2010), no. 4, 1393–1428. http://projecteuclid.org/euclid.ijm/1348505534.

[3] N. Boros, P. Janakiraman, A. Volberg, *Perturbation of Burkholder's martingale transform and Monge–Ampère equation* Adv. Math., 230 (4–6) (2012), pp. 2198?2234

[4] P. Ivanishvili, N. Osipov, D. Stolyarov, V. Vasyunin, P. Zatitskiy, *Bellman function for extremal problems in BMO,* to appear in Transactions of the Amer. Math. Soc. 2013

[5] P. Ivanisvili, *Burkholder's martingale transform*, 2014, http://arxiv.org/abs/1402.4751

[1] Value of $C(\beta')$ for $\beta' \in (-1+2/p, s_0)$ was hard to express in simple way. Implicit expression of this value is given in [5], Theorem 2, part (ii).