37. Cars cross a certain point in the highway in accordance with a Poisson process with rate \( \lambda = 3 \) per minute. If Reb runs blindly across the highway, then what is the probability that she will be uninjured if the amount of time that it takes her to cross the road is \( s \) seconds? (Assume that if she is on the highway when a car passes by, then she will be injured.) Do it for \( s = 2, 5, 10, 20 \).

**Note:** Only given for context.

38. Suppose in Exercise 37 that Reb is agile enough to escape from a single car, but if she encounters two or more cars while attempting to cross the road, then she will be injured. What is the probability that she will be unhurt if it takes her \( s \) seconds to cross? Do it for \( s = 5, 10, 20, 30 \).

**Solution:** Reb is uninjured if and only if she encounters zero or one car while crossing the road. Let \( N_t \) denote the number of cars she has encountered during the crossing. Then

\[
P\{\text{Reb is uninjured}\} = P\{N_s = 0 \text{ or } N_s = 1\} \\
= P\{N_s = 0\} + P\{N_s = 1\} \\
= e^{-\lambda s}(\lambda s)^0 + e^{-\lambda s}(\lambda s)^1 \\
= e^{-\lambda s}(1 + \lambda s).
\]

Lastly, note that because \( \lambda \) is in units \( \text{min}^{-1} \) and \( s \) is in seconds, we need to convert:

\[
\lambda = \frac{3 \text{ min}}{60 \text{ sec}} = \frac{1}{20} \text{ sec}^{-1},
\]

and the answer is

\[
e^{-\frac{s}{20}} \left(1 + \frac{s}{20}\right).
\]

Plugging in 5, 10, 20, and 30 seconds, the probabilities are (approximately) 0.9735, .9098, .7358, and .5578, respectively.

42. Let \( \{N(t), t \geq 0\} \) be a Poisson process with rate \( \lambda \). Let \( S_n \) denote the time of the \( n \)th event. Find

(a) \( E[S_4] \)

**Solution:**

\[
\frac{4}{\lambda}
\]

(b) \( E[S_4|N(1) = 2] \)

**Solution:**

\[
E[S_4|N_1 = 2] = E[S_4|T_2 \geq 1] = E[1 + S_2] = 1 + \frac{2}{\lambda}
\]
(c) $E[N(4) - N(2) | N(1) = 3]$

**Solution:** We only care about the difference in expected values. Recall that all increments are independent, and so the number of arrivals $N_4 - N_2$ in $[2, 4]$ does not depend upon any prior times or intervals. Thus,

$$E[N_4 - N_2 | N_1 = 3] = E[N_4 - N_2] = 4\lambda - 2\lambda = 2\lambda.$$

51. If an individual has never had a previous automobile accident, then the probability he or she has an accident in the next $h$ time units is $\beta h + o(h)$; on the other hand, if he or she has never had a previous accident, then the probability is $\alpha h + o(h)$. Find the expected number of accidents an individual has by time $t$.

**Solution:** First, note that the “little-oh” notation tells you that these are Poisson processes.

Let $T$ be the time of the first accident, and let $N_t$ be the number of accidents by time $t$. Let $M_t$ be the number of accidents after time $T$. We’ll condition on $T$, using the fact that it is exponential with rate $\beta$:

$$E[N_t] = \int_0^\infty E[N_t | T = u] \beta e^{-\beta u} du$$

$$= \int_0^t E[N_t | T = u] \beta e^{-\beta u} du + \int_t^\infty E[N_t | T = u] \beta e^{-\beta u} du$$

$$= \int_0^t E[N_t | T = u] \beta e^{-\beta u} du + \int_t^\infty 0 \cdot \beta e^{-\beta u} du,$$

where the last line is reasoned as follows: if $u \in (t, \infty)$, then $t < u$, and so no accidents have happened yet.

Continuing,

$$E[N_t] = \int_0^t E[N_t | T = u] \beta e^{-\beta u} du$$

$$= \int_0^t E[1 + M_{t-u}] \beta e^{-\beta u} du$$

$$= \int_0^t (1 + \alpha(t - u)) \beta e^{-\beta u} du$$

$$= (1 + \alpha t) (1 - e^{-\beta t}) + \alpha te^{-\beta t} - \frac{\alpha}{\beta} (1 - e^{-\beta t})$$

$$= \left(1 + \alpha t - \frac{\alpha}{\beta}\right) (1 - e^{-\beta t}) + \alpha te^{-\beta t}.$$

As a quick check, notice that if $\alpha = \beta$, then we have the same Poisson process for the entire time. In such a case, the expected value should be $\beta t$. Indeed, in our expression, we have

$$E[N_t] = (1 + \beta t - 1) (1 - e^{-\beta t}) + \beta te^{-\beta t}$$

$$= \beta t - \beta e^{-\beta t} + \beta te^{-\beta t} = \beta t.$$