43. Customers arrive at a two-server service station according to a Poisson process with rate $\lambda$. Whenever a new customer arrives, any customer that is in the system immediately departs. A new arrival enters service first with server 1 and then with server 2. If the service times at the servers are independent exponentials with respective rates $\mu_1$ and $\mu_2$, what proportion of entering customers completes their service with server 2?

**Solution:** Let $X_1$ and $X_2$ denote the time the customer spends at servers 1 and 2, respectively, and let $T$ denote the time of the next customer arrival. Then we seek to compute $P(X_1 + X_2 < T)$:

$$P(X_1 + X_2 < T) = \int_0^\infty P(x_1 + X_2 < T) f_{X_1}(x_1) \, dx_1$$

$$= \int_0^\infty \int_0^\infty P(x_1 + x_2 < T) f_{X_1}(x_1) f_{X_2}(x_2) \, dx_1 \, dx_2$$

$$= \int_0^\infty \int_0^\infty e^{-\lambda(x_1+x_2)} \mu_1 e^{-\mu_1 x_1} \mu_2 e^{-\mu_2 x_2} \, dx_1 \, dx_2$$

$$= \mu_1 \mu_2 \left( \int_0^\infty e^{-(\lambda+\mu_1)x_1} \, dx_1 \right) \cdot \left( \int_0^\infty e^{-(\lambda+\mu_2)x_2} \, dx_2 \right)$$

$$= \mu_1 \mu_2 \left( \frac{1}{\lambda + \mu_1} \right) \cdot \left( \frac{1}{\lambda + \mu_2} \right)$$

$$= \frac{\mu_1 \mu_2}{\lambda^2 + \lambda \mu_1 + \lambda \mu_2 + \mu_1 \mu_2}.$$

Notice that we can rewrite this as

$$\frac{1}{\left(1 + \frac{\lambda}{\mu_1}\right) \left(1 + \frac{\lambda}{\mu_2}\right)}.$$

which approaches 1 as $\mu_1, \mu_2 \to \infty$. This makes sense, because if service speed is high, then the chance of service being completed before another customer arrives increases.

45. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda$, that is independent of the nonnegative random variable $T$ with mean $\mu$ and variance $\sigma^2$. Find

(a) $\text{Cov} (T, N(T))$

**Solution:** Use this gem of a rule:

$$\text{Cov} (X, Y) = \text{Cov} (X, E [Y|X]).$$

In this case,

$$\text{Cov} (T, N(T)) = \text{Cov} (T, E [N(T)|T]) = \text{Cov} (T, \lambda T) = \text{Cov} (T, T) = \lambda \text{Var} (T) = \lambda \sigma^2.$$
Solution: We'll just compute it directly:

\[ E[N(T)] = E[E[N(T)|T]] = E[\lambda T] = \lambda E[T]. \]

Also, because \( \text{Var}(N_t) = \lambda t = E[N_t^2] - (E[N_t])^2 \), we have \( E[N_t^2] = \lambda t + \lambda^2 t^2 \). Therefore,

\[ E[N^2(T)] = E[E[N^2(T)|T]] = E[\lambda^2 T^2] = \lambda^2 E[T^2]. \]

Hence,

\[ \text{Var}(N(T)) = E[N^2(T)] - (E[N(T)])^2 = \lambda^2 E[T^2] - \lambda^2 (E[T])^2 \]

\[ = \lambda^2 \text{Var}(N(T)) = \lambda^2 \sigma^2. \]

47. Consider a two-server parallel queueing system where customers arrive according to a Poisson process with rate \( \lambda \), and where the service times are exponential with rate \( \mu \). Moreover, suppose that arrivals finding both servers busy immediately depart without receiving service (such a customer is said to be lost), whereas those finding at least one free server immediately enter service and then depart when their service is completed.

(a) If both servers are presently busy, find the expected time until the next customer enters the system.

Solution: Let \( T \) be the time before the next customer enters service. Let \( Y_1 \) and \( Y_2 \) be the service times for customers 1 and 2, respectively, and let \( Y = \min(Y_1, Y_2) \). Note that because \( Y_1 \) and \( Y_2 \) are exponential with rate \( \mu \), then \( Y \) is also exponential with rate \( 2\mu \). Let \( X \) be the arrival time of the next customer. If \( Y < X \), then at least one server is open, and so \( T = X \). Otherwise, we have to start the process over, and add \( X \) to the total time. Therefore,

\[ E[T] = E[T \mathbb{1}_{\{Y < X\}} + T \mathbb{1}_{\{Y \geq X\}}] \]

\[ = E[X] P(Y < X) + E[X + T] P(Y \geq X) \]

\[ = E[X] P(Y < X) + E[X] P(Y > X) + E[T] P(Y \geq X) \]

\[ = E[X] + E[T] P(Y \geq X). \]

Thus,

\[ E[T] = \frac{E[X]}{1 - P(Y \geq X)} = \frac{E[X]}{P(Y < X)} = \frac{1}{\frac{\lambda}{2\mu + \lambda}} = \frac{1}{\lambda} + \frac{1}{2\mu}. \]

(b) Starting empty, find the expected time until both servers are busy.

Solution: Let \( T_2 \) denote the time until both servers are busy, and let \( X_1 \) and \( X_2 \) be the interarrival times of the first two customer arrivals, and let \( Y_1 \) and \( Y_2 \) be the service times of the two customers, respectively. We must wait at least until time \( X_1 \) for the first customer to arrive. After that, either (a) the next customer arrives before the first customer finishes \( (X_2 < Y_1) \), in which case \( T_2 = X_1 + X_2 \); or (b) the first customer finishes before next
customer arrives \((X_2 > Y_1)\), in which case we restart the process at time \(X_1 + X_2\). Thus,

\[
E[T_2] = E[T_2 \mathbb{1}_{X_2 < Y_1} + T_2 \mathbb{1}_{X_2 \geq Y_1}]
\]

\[
= E[(X_1 + X_2) \mathbb{1}_{X_2 < Y_1} + (X_1 + X_2 + T_2) \mathbb{1}_{X_2 \geq Y_1}]
\]

\[
= E[(X_1 + X_2) (\mathbb{1}_{X_2 < Y_1} + \mathbb{1}_{X_2 \geq Y_1}) + T_2 \mathbb{1}_{X_2 \geq Y_1}]
\]

\[
= E[(X_1 + X_2) (1)] + E[T_2 \mathbb{1}_{X_2 \geq Y_1}]
\]

\[
= E[X_1] + E[X_2] + E[T_2]P(X_2 \geq Y_1),
\]

and so

\[
E[T_2] = \frac{E[X_1] + E[X_2]}{1 - P(X_2 \geq Y_1)} = \frac{\frac{1}{\lambda} + \frac{1}{\mu}}{\frac{2(\mu + \lambda)}{\lambda + \mu}} = \frac{2(\mu + \lambda)}{\lambda} = \frac{2(\mu + \lambda)}{\lambda^2}.
\]

(Note that this solution assumes that at most one customer leaves before the next customer arrives. Think about what assumptions this makes on the relative magnitudes of \(\lambda\) and \(\mu\).)

(c) Find the expected time between two successive lost customers.

**Solution:** Assume that the last customer was lost (so both servers are currently occupied). Let \(T\) be the time until the last customer was lost, let \(X_1\) be the time until the next customer arrives, and let \(Y_1\) and \(Y_2\) be the service times of the current two customers.

We have a few cases to consider: (a) the next customer arrives before either of the current customers finishes \((X_1 < Y = \min(Y_1, Y_2))\), in which case \(T = X_1\); (b) the customer arrives after one of the current customers is finished, but not both \((Y < X_1 < Y = \max(Y_1, Y_2))\), in which case we start over at time \(X_1\); or (c) both customers finish before the next arrival \((\max(Y_1, Y_2) < X_1)\), in which case we need to wait for \(T_2\) from the previous problem and then start over.

Therefore:

\[
E[T] = E[T \mathbb{1}_{X_1 < Y} + T \mathbb{1}_{Y < X_1 < Y} + T \mathbb{1}_{Y < X_1}]
\]

\[
= E[(X_1) \mathbb{1}_{X_1 < Y} + (X_1 + T) \mathbb{1}_{Y < X_1 < Y} + (X_1 + T_2 + T) \mathbb{1}_{Y < X_1}]
\]

\[
= E[X_1]P(X_1 < Y) + E[X_1 + T]P(Y < X_1 < Y)
\]

\+

\[
E[X_1 + T_2 + T]P(Y < X_1)
\]

\[
= E[X_1] (P(X_1 < Y) + P(Y < X_1 < Y) + P(Y < X_1))
\]

\+

\[
+ E[T] (P(Y < X_1 < Y) + P(Y < X_1)) + E[T_2]P(Y < X_1)
\]

\[
= E[X_1] + E[T]P(Y < X_1) + E[T_2]P(Y < X_1)
\]

Next, without loss of generality, suppose that we’ve ordered \(Y_1\) and \(Y_2\) such that \(Y_1 < Y_2\). Then \(Y = Y_2\), and

\[
E[T] = E[X_1] + E[T]P(Y < X_1) + E[T_2]P(X_1 > Y_2),
\]
whereby

\[
E[T] = \frac{E[X_1] + E[T_2]P(X_1 > Y_2)}{1 - P(Y < X_1)}
\]

\[
= \frac{\frac{1}{\lambda} + \frac{2(\mu+\lambda)}{\lambda^2} \frac{\mu}{\lambda+\mu}}{P(Y > X_1)}
\]

\[
= \frac{\frac{1}{\lambda} + \frac{2(\mu+\lambda)}{\lambda^2} \frac{\mu}{\lambda+\mu}}{\frac{\lambda}{\lambda+2\mu}}
\]

\[
= \frac{(\lambda + 2\mu)^2}{\lambda^3}.
\]

Note that the numerator has units of time\(^{-2}\), and the denominator has units of time\(^{-3}\), for net units time\(^1\), which is what we need.

Also, notice that we can rewrite this as

\[
\frac{1}{\lambda} \left(\frac{\lambda + 2\mu}{\lambda}\right)^2 = \frac{1}{\lambda} \left(1 + \frac{2\mu}{\lambda}\right)^2
\]

Suppose that \(\mu \ll \lambda\), so that the customer arrival rate is much greater than the service rate. Then the expected time approaches

\[
\frac{1}{\lambda} (1 + 0)^2 = \frac{1}{\lambda},
\]

that is, the expected time between “bounced” customers is equal to the expected time between arrivals, i.e., all customers are lost, just as we would expect by intuition.

53. The water level of a certain reservoir is depleted at a constant rate of 1000 units daily. The reservoir is refilled by randomly occurring rainfalls. Rainfalls occur according to a Poisson process with rate 0.2 per day. The amount of water added to the reservoir by a rainfall is 5000 units with probability 0.8 or 8000 units with probability 0.2. The present water level is slightly below 5000 units.

(a) What is the probability the reservoir will be empty after five days?

**Solution:** The reservoir begins with 5000 units and loses 1000 units per day. Thus, if there is no rainfall, it runs out of water at precisely 5 days. Hence, the reservoir is empty after five days if and only if there are no rainfalls, i.e.,

\[
P(N_5 = 0) = e^{-0.2 \cdot 5} = e^{-1} \approx 36.79\%.
\]

(b) What is the probability the reservoir will be empty sometime within the next ten days?

**Solution:** Let \(N_t\) be the number of rainfalls by time \(t\), and let \(T_i\) be the interarrival time of the \(i^{th}\) rainfall. By part (a), the reservoir will go empty prior to time 5 if and only if there are zero rainfalls in \([0, 5)\). Suppose there is a rainfall at just before time 5, so that the reservoir is not empty prior to time 5. If the rainfall is 8000 units at time \(T_1 < 5\), then the water satisfies

\[
W_t = 13000 - 1000t,
\]
and so even with no additional rainfalls, the reservoir cannot run empty prior to time 10. (Set equal to 0 and solve for \( t \).)

If the rainfall is 5000 units at time \( T_1 < 5 \), then the water satisfies

\[ W_t = 10000 - 1000t, \]

and the reservoir goes empty at time 10 (or just before that time, since we start with “just under” 5000 units) without additional rainfall. Therefore,

\[
P( \text{reservoir goes empty before 10 days}) = P( \text{reservoir has no rain before 5 days})
\]

\[
+ P( \text{reservoir has 5000 units of rain at } T_1 < 5 \text{ and no rain after } T_1)
\]

\[
= e^{-0.25}
\]

\[
+ P( \text{reservoir has 5000 units of rain at } T_1 < 5)P(\text{no rain after } T_1)
\]

\[
= e^{-0.25} + 0.8 \cdot P(\text{no rain after } T_1)
\]

\[
= e^{-1} + 0.8 \cdot P(N_{10} = 1)
\]

\[
= e^{-1} + 0.8e^{-0.2 \cdot 10}(0.2 \cdot 10)
\]

\[
= e^{-1} + 0.8e^{-2}(2) = e^{-1} + 1.6e^{-2} \approx 58.44\%.
\]