1. Suppose that cars arrive at a stoplight (from one of four directions) in a Poisson process at a rate of 10 cars per minute. Denote the number of cars that have arrived after $s$ minutes by $N_s$.

(a) (1 point) Write the probability that $n$ cars arrive between time $t$ and time $t + s$.

Solution: The rate is 10 cars per minute, and the time is in minutes, so no conversion is required. Thus,

$$P \{ N_{t+s} - n_t = n \} = e^{-10s}(10s)^n \frac{n!}{n!}$$

(b) (1 point) Suppose that a red light lasts for $s$ minutes. What is the expected number of cars to be stopped at the intersection at the end of the light?

Solution: $10s$ minutes.

(c) (3 points) Suppose that a local radio station will give a prize to a listener if they see (exactly) 40 cars at the stoplight at the end of a red light. For any one red light, what is the probability that a listener can win the prize?

Solution:

$$P \{ N_s = 40 \} = e^{-10s}(10s)^{40} \frac{40!}{40!}$$

(d) (2 points) Suppose that a red light lasts 2 minutes, and the expected number of cars are stopped at the end of the red light. (See part (b).) Assume that the time for each car to clear the intersection after the red light is exponentially distributed with mean time 12 seconds = 0.2 minutes. What is the expected time required for all the expected cars to clear the intersection?

Solution: There are $10 \cdot 2 = 20$ cars expected. Let $T_n$ denote the time for the $n^{th}$ car to clear the intersection. Then

$$E [T_1 + \cdots + T_{20}] = \sum_{n=1}^{20} E[T_n] = \sum_{n=1}^{20} 0.2 = 4 \text{ minutes.}$$

(e) (3 points) What is the probability density function for the time for all the expected cars to clear the intersection? Use this probability density function to set up (but not evaluate) the probability that more than 5 minutes are required for all the cars to clear the intersection.

Solution: This problem was longer than I originally thought when I wrote it. Sorry!
Let $S = T_1 + \cdots + T_{20}$ be the total time for all the cars to clear the intersection. We know that this has a gamma density function

$$f_S(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{19}}{19!}, (t \geq 0)$$

where $\lambda$ is a rate we must determine. By the known exponential process, for each $n$,

$$E[T_n] = \frac{1}{\lambda} = 0.2 \text{ min} \implies \lambda = 5 \text{ min}^{-1}.$$ 

Thus,

$$f_S(t) = 5e^{-5t} \frac{(5t)^{19}}{19!}, (t \geq 0),$$

and

$$P\{S > 5\} = \int_{5}^{\infty} 5e^{-5t} \frac{(5t)^{19}}{19!} dt.$$ 

By the way, you can do some reasoning to see that if we define $M_t$ to be the number of cars that have cleared the intersection by time $t$,

$$P(S > 5) = P(M_5 < 20) = \sum_{n=0}^{19} e^{-25} \frac{(25)^{19}}{19!} \approx 13.4\%.$$