Let $B(t)$ denote Brownian motion, and let $V_t = e^{-t}B(e^{2t})$ denote the Ornstein-Uhlenbeck (OU) process.

1. (3 points) Calculate the covariance function $(s, t) \rightarrow \mathbb{E}[V_t V_s] = \text{Cov}(V_t, V_s)$ for the OU process, where $0 \leq s \leq t$. (Hint: Recall that $\text{Cov}(B(t), B(s)) = s$ for all $s \leq t$.)

Solution:

$$
\text{Cov}(V_s, V_t) = \text{Cov}(e^{-s}B(e^{2s}), e^{-t}B(e^{2t}))
= e^{-s}e^{-t}\text{Cov}(B(e^{2s}), B(e^{2t}))
= e^{-s}e^{-t}e^{2s}
= e^{s-t} = e^{-t-s}.
$$

2. (2 points) What is the distribution for the OU process? (Hint: What are the mean and variance for the OU process, given the previous problem?)

Solution: We already know the mean is 0. (Or directly, $\mathbb{E}[V_t] = \mathbb{E}[e^{-t}B(e^{2t})] = e^{-t}\mathbb{E}[B(e^{2t})] = e^{-t} \cdot 0$). As for the variance, by part (a),

$$
\text{Var}(V_t) = \text{Cov}(V_t, V_t) = e^{-(t-t)} = e^0 = 1.
$$

Therefore, $V_t \sim \mathcal{N}(0, 1)$.

3. (5 points) Let $\vec{V}_t = (V_t, V_1)$ be a centered Gaussian vector, where $V_t$ is the OU process. Compute the covariance matrix $\sigma^2$ for this CGV, as well as $\det(\sigma^2)$ and $(\sigma^2)^{-1}$.

Solution: I’m getting soft in my old age. :-) The covariance matrix:

$$
\sigma^2 = \begin{bmatrix}
\text{Cov}(V_t, V_t) & \text{Cov}(V_t, V_1) \\
\text{Cov}(V_1, V_t) & \text{Cov}(V_1, V_1)
\end{bmatrix}
= \begin{bmatrix}
1 & e^{-(1-t)} \\
e^{-(1-t)} & 1
\end{bmatrix}.
$$

Next, the determinant is

$$
1 \cdot 1 - e^{-(1-t)} \cdot e^{-(1-t)} = 1 - e^{-2(1-t)} = 1 - \alpha^2(t),
$$

where

$$
\alpha(t) = e^{-(1-t)}.
$$

Lastly, the inverse is

$$
(\sigma^2)^{-1} = \frac{1}{1 - \alpha^2(t)} \begin{bmatrix}
1 & -\alpha(t) \\
-\alpha(t) & 1
\end{bmatrix}.
$$