Section 1.4 Problems

9. Solve the given initial value problem, and determine the interval of existence of the solution.

\[ \frac{dy}{dt} = \frac{3t^2 + 4t + 2}{2(y-1)} \quad y(0) = -1 \]

**Solution:** (15 points) This is a first-order, nonlinear problem, so we use separation of variables:

\[
(y-1)y' = \frac{1}{2} (3t^2 + 4t + 2) \]

\[
\Rightarrow \int_{t_0}^{t} (y(s)-1)y'(s) \, ds = \frac{1}{2} \int_{t_0}^{t} (3s^2 + 4s + 2) \, ds
\]

\[
\Rightarrow \int_{y_0}^{y} (r-1) \, dr = \frac{1}{2} \int_{t_0}^{t} (3s^2 + 4s + 2) \, ds
\]

\[
\Rightarrow \int_{-1}^{y} (r-1) \, dr = \frac{1}{2} \int_{0}^{t} (3s^2 + 4s + 2) \, ds
\]

\[
\Rightarrow \frac{1}{2} y^2 - y - \frac{1}{2} - 1 = \frac{1}{2} (t^3 + 2t^2 + 2t)
\]

\[
\Rightarrow y = 1 \pm \sqrt{t^3 + 2t^2 + 2t + 4}.
\]

To choose the sign of the root, we note that in the expression above, \( y(0) \) can only be negative if the root is negative. So, the solution is

\[ y(t) = 1 - \sqrt{t^3 + 2t^2 + 2t + 4}. \]

For interval of existence, we seek times \( t_{\text{start}} < t_0 = 1 < t_{\text{end}} \) where the solution fails, such that the solution never fails on \( (t_{\text{start}}, t_{\text{end}}) \). Well, for this solution, the only way it can fail is by taking the square root of a negative number, so we find roots of

\[ t^3 + 2t^2 + 2t + 4 = 0. \]

The quickest way to do this is to factor as below:

\[ t^3 + 2t^2 + 2t + 4 = t^2(t + 2) + 2(t + 2) = (t + 2)(t^2 + 2), \]

so the only real root is \( t = -2 \).

Alternatively, use the rational root theorem to guess a first root: if there’s a rational root, it’s one of \( \{ \frac{\pm 1}{1}, \frac{\pm 2}{1}, \frac{\pm 4}{1} \} \). Evaluation of these shows that \( -2 \) is a root, and we can factor it (by long division) to

\[ (t + 2)(t^2 + 2). \]

The other two roots are complex. So, the solution is valid on \( (-2, \infty) \).
15. Solve the given initial value problem.

\[
\frac{dy}{dt} = y + \sqrt{t^2 + y^2} \quad y(1) = 0
\]

**Solution: (10 points)** This is a first-order, nonlinear problem, so we need separation of variables. However, the problem is not immediately separable. (When you have factors like \( t \pm y \) or \( t^2 \pm y^2 \), it’s a good hint that this is the case.) We know that we’d eventually like to have \( f(y)y' = g(t) \), so dividing both sides by \( t \) is a good start:

\[
y' = \frac{y}{t} + \frac{1}{t} \sqrt{t^2 + y^2}
\]

which suggests a new variable \( v = \frac{y}{t} \). Then

\[
y(t) = tv(t) \implies y'(t) = v(t) + tv'(t),
\]

and

\[
v + tv' = v + \sqrt{1 + v^2} \implies tv' = \sqrt{1 + v^2} \implies \frac{v'}{\sqrt{1 + v^2}} = \frac{1}{t} \implies \int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{1}{t} dt = \ln |t| + c.
\]

To integrate the left side, we use a trig substitution:

\[
\sqrt{1 + v^2} \theta
\]

Notice

\[
v = \tan \theta \implies dv = \sec^2 \theta d\theta,
\]

and the integral is

\[
\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta|
\]

\[
= \ln \left| \sqrt{1 + v^2} + v \right|.
\]

To remember how to do that integral, just look it up in a table, or see this document on my website from an old calculus class:

It’s easier to plug in the initial condition now: \( v(1) = \frac{y(1)}{1} = 0: \)

\[
\ln \left| \sqrt{1 + 0} \right| = \ln |1| + c \implies c = 0.
\]

Thus, when we plug in \( v = \frac{y}{t} \) and solve for \( y \), we have

\[
\left| \sqrt{1 + \left(\frac{y}{t}\right)^2 + \frac{y}{t}} \right| = |t| \implies \left| \sqrt{t^2 + y^2 + y} \right| = t^2.
\]

Now, notice that from the original differential equation, if \( y \geq 0 \) and \( t > 0 \),

\[
y' = \frac{1}{t} \left( y + \sqrt{t^2 + y^2} \right) > 0.
\]

So, if the solution starts at or above 0, it stays above zero. That’s the case here, as \( y(1) = 0 \). So,

\[
t^2 = \left| \sqrt{t^2 + y^2 + y} \right| = \sqrt{t^2 + y^2} + y,
\]

and we can solve for \( y \):

\[
y(t) = \frac{t^2 - 1}{2}.
\]

**Completeness Points**

(4 points) For problems 7, 11, 17, and 19, \( \frac{1}{2} \) point per moderately-attempted problem, and another \( \frac{1}{2} \) point per very-seriously, nearly-properly-solved problem.