Solve the following problems. Try to solve each problem within the suggested time.

1. (3 minutes) Find the general solution of

\[ 2 \frac{dy}{dt} + (\cos t) y = \cos t. \]

**Solution:** First, note that this is a first-order, linear, non-homogeneous problem, so we use an integrating factor. However, the equation must be in the form

\[ y' + f(t)y = g(t), \]

and this equation is not. So, we divide by 2:

\[ y' + \frac{1}{2} \cos t y = \frac{1}{2} \cos t. \]

For the integrating factor \( \mu \):

\[
(\mu y)' = \mu y' + \left( \frac{1}{2} \cos t \mu \right) y = \frac{1}{2} \mu \cos t
\]

\[ = \mu' \]

\[ \implies \mu' - \frac{1}{2} \cos t \mu = 0 \]

\[ \implies \mu = e^{- \int -\frac{1}{2} \cos t dt} = e^{\frac{1}{2} \sin t}. \]

So,

\[ (\mu y)' = \frac{1}{2} \cos t e^{\frac{1}{2} \sin t} \implies \mu y = e^{\frac{1}{2} \sin t} + c \]

\[ \implies y = 1 + ce^{-\frac{1}{2} \sin t}. \]

2. (7 minutes) Find the general solution of

\[ y + \left( x + \frac{1}{1 + y^2} \right) \frac{dy}{dx} = -17 \cos x \]

**Solution:** This is a first-order, nonlinear problem, so we need either separation of variables, a new variable (followed by separation of variables), or the exact equation method. The mixture of terms multiplying the \( y' \) term doesn’t look promising for separation of variables. Dividing by \( x \) doesn’t do much good, either, so a new variable is probably out. So, we’ll try the exact equation method.

Notice, however, that this equation is not in the correct form; the righthand side should be zero. So first, we fix that and identify \( M \) and \( N \):

\[
\underbrace{y + 17 \cos x}_M + \underbrace{\left( x + \frac{1}{1 + y^2} \right)}_N y' = 0.
\]
Let's verify that the equation is exact:

\[ M_y = 1, \text{ and } N_x = 1. \]

These are equal, so the equation is exact.

Now, the equations for \( \varphi \) are:

\[
\begin{align*}
\varphi_x & = M = y + 17 \cos x \\
\varphi_y & = N = x + \frac{1}{1 + y^2}.
\end{align*}
\]

By the first equation,

\[
\begin{align*}
\varphi & = \int \varphi_x \, dx \\
& = \int (y + 17 \cos x) \, dx \\
& = xy + 17 \sin x + c(y).
\end{align*}
\]

So,

\[
N = x + \frac{1}{1 + y^2} = \varphi_y \\
\quad = \frac{\partial}{\partial y} (xy + 17 \sin x + c(y)) \\
\quad = x + c'(y),
\]

whereby

\[
c'(y) = \frac{1}{1 + y^2} \implies c(y) = \arctan y + d.
\]

So,

\[
\varphi(x, y) = xy + 17 \sin x + \arctan y + d,
\]

and this gives the solution to the original differential equation by setting it equal to a constant (or zero, since we already have a constant):

\[
xy + 17 \sin x + \arctan y = d.
\]