Solve the following problems. Try to solve each problem within the suggested time.

1. (10 minutes) Find the general solution of

\[ 2y'' + y' - 6y = 7te^{-2t}. \]

**Solution:** First, we find the solutions to the homogeneous problem:

\[ 2r^2 + r - 6 = (2r - 3)(r + 2) = 0, \]

and so \( y_1(t) = e^{-2t} \) and \( y_2(t) = e^{\frac{3}{2}t}. \) We need a particular solution \( \psi. \)

**Method 1 (Judicious Guessing):** We guess that \( \psi = pe^{-2t}, \) where \( p \) is a polynomial of degree \( m. \) So,

\[
\begin{align*}
\psi &= e^{-2t}p \\
\psi' &= e^{-2t}(p' - 2p) \\
\psi'' &= e^{-2t}(p'' - 4p' + 4p),
\end{align*}
\]

whereby

\[
\begin{align*}
2\psi'' &= e^{-2t}(2p'' - 8p' + 8p) \\
+ \psi' &= e^{-2t}(p' - 2p) \\
-6\psi &= e^{-2t}(-6p) \\
7te^{-2t} &= e^{-2t}(2p'' - 7p')
\end{align*}
\]

Thus,

\[ 7t = 2p'' - 7p'. \]

The lefthand side is a polynomial of degree 1, and the righthand side is a polynomial of degree \( m - 1. \) (\( p'' \) has degree \( m - 2 \), and \( p' \) has degree \( m - 1. \) The highest power of \( t \) is \( m - 1. \)) Thus, \( 1 = m - 1, \) i.e., \( m = 2, \) and

\[ p(t) = at^2 + bt + c. \]

Plugging this into our equation for \( p, \)

\[ 7t = 2(2a) - 7(2at + b) \implies (7)t + 0 = (-14a)t + (4a - 7b). \]

So,

\[
\begin{align*}
7 &= -14a \implies a = -\frac{1}{2} \\
0 &= 4a - 7b \implies b = \frac{4}{7}a = -\frac{2}{7}.
\end{align*}
\]

Therefore, the general solution is

\[ y(t) = c_1e^{-2t} + c_2e^{\frac{3}{2}t} - e^{-2t}\left(\frac{1}{2}t^2 + \frac{2}{7}t\right). \]
Method 2 (Variation of Parameters): We seek a solution of the form \( \psi = u_1y_1 + u_2y_2 \). First, we note that the equation is not in the form \( y'' + py' + qy = g \), so we fix that:

\[
y'' + \frac{1}{2}y' - 3y = \frac{7}{2}te^{-2t}.
\]

It wouldn't be wrongski to compute the Wronskian next:

\[
W[y_1, y_2] = \det \left( \begin{array}{cc} e^{-2t} & e^{\frac{3}{2}t} \\ -2e^{-2t} & \frac{3}{2}e^{\frac{3}{2}t} \end{array} \right) = \frac{7}{2}e^{-\frac{3}{2}t}.
\]

So,

\[
u_1' = -\frac{gy_2}{W} = -t \implies u_1 = -\frac{1}{2}t^2,
\]

and

\[
u_2' = \frac{gy_1}{W} = te^{-\frac{3}{2}t} \implies u_2 = -\frac{2}{7}te^{-\frac{3}{2}t} - \frac{4}{49}e^{-\frac{3}{2}t}.
\]

Thus,

\[
\psi = u_1y_1 + u_2y_2 = e^{-2t} \left( -\frac{1}{2}t^2 - \frac{2}{7}t - \frac{4}{49} \right),
\]

and the general solution is

\[
y(t) = c_1y_1 + c_2y_2 + \psi
\]

\[
= c_1e^{-2t} + c_2e^{\frac{3}{2}t} - e^{-2t} \left( \frac{1}{2}t^2 + \frac{2}{7}t + \frac{4}{49} \right)
\]

\[
= c_1e^{-2t} + \left( c_2 - \frac{4}{49} \right) e^{\frac{3}{2}t} - e^{-2t} \left( \frac{1}{2}t^2 + \frac{2}{7}t \right)
\]

\[
= c_1e^{-2t} + \tilde{c}_2e^{\frac{3}{2}t} - e^{-2t} \left( \frac{1}{2}t^2 + \frac{2}{7}t \right).
\]