Math 3D - Additional “Judicious Guessing” Examples
Paul Macklin
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The method of judicious guessing can be tricky to get the hang of, but it is rewarding. It can save you a lot of work when compared to variation of parameters, even in the longer examples.

Here, I hope to help clarify some additional cases of “judicious guessing”. I’m going to work several of the odd exercises in your text (page 164), as well as problem 10, the “stumper problem”. Many thanks to those who contributed ideas on how to solve that problem!

Exercises
Find a particular solution to each of the following equations.

1. \(y'' + 3y = t^3 - 1\)

**Solution:** Because the right-hand side is a polynomial, we expect \(\psi\) to be of the form of a polynomial. Next, notice that the solutions to the homogeneous problem are \(\cos(\sqrt{3}t)\) and \(\sin(\sqrt{3}t)\), and so neither \(t^3\) nor 1 is a solution to the homogeneous DE. So, we only expect to need a 3rd degree polynomial:

\[
\psi(t) = a + bt + ct^2 + dt^3
\]

\[
\psi'(t) = b + 3ct + 3dt^2
\]

\[
\psi''(t) = 3c + 6dt,
\]

and so

\[
\psi''(3) + 3\psi + 3t^2\psi \quad t^3(1) + t^2(0) + t(0) + 1(-1) = t^3(3d) + t^2(3c) + t(3b + 6d) + 1(3c + 2a).
\]

By equating coefficients of these polynomials, we get

\[
d = \frac{1}{3}, c = 0, b = -\frac{2}{3}, a = -\frac{1}{3},
\]

and so

\[
\psi(t) = -\frac{1}{3} - \frac{2}{3}t + \frac{1}{3}t^3.
\]

3. \(y'' - y = t^2e^t\)

**Solution:** Because the right-hand side is a polynomial times an exponential, we expect a particular solution of the form

\[
\psi(t) = v(t)e^t
\]

\[
\psi'(t) = v'e^t + ve^t
\]

\[
\psi''(t) = v''e^t + 2v'e^t + ve^t
\]

where \(v\) is a polynomial of degree at least 2. Let’s plug this in to get a better feeling for what degree \(v\) needs to be.

\[
\psi'' - \psi = e^t(\psi'' + 2v' + v)
\]

\[
e^t \left( t^2(1) + t(0) + 1(0) \right) = e^t \left( v'' + 2v' \right)
\]

(1)
Notice that if the degree of \( v \) is \( m \), then the degree of \( v'' + 2v' \) is \( m - 1 \). Because we need this to be a second-degree polynomial, we need \( m - 1 = 2 \), i.e., \( m = 3 \). Therefore,

\[
\psi = v(t)e^t = e^t(a + bt + ct^2 + dt^3).
\]

If we plug in this refined guess (into (1), we get

\[
e^t(t^2) = e^t\left(2c + 6dt + 2b + 4ct + 6dt^2\right) = e^t\left(t^2(6d) + t(6d + 4c) + 1(2c + 2b)\right).
\]

By equating coefficients, we have

\[
d = \frac{1}{6}, c = -\frac{1}{4}, b = \frac{1}{4},
\]

and \( a \) is undetermined. Therefore, we get a whole family of solutions

\[
\psi = e^t\left(a + \frac{1}{4}t - \frac{1}{4}t^2 + \frac{1}{6}t^3\right).
\]

We might as well pick the simplest solution and set \( a = 0 \).

\[
\psi = e^t\left(\frac{1}{4}t - \frac{1}{4}t^2 + \frac{1}{6}t^3\right).
\]

5. \( y'' + 2y' + y = e^{-t} \)

**Solution:** Because the right-hand side is an exponential, we might expect a particular solution of the form \( ce^{-t} \). However, we see that both \( e^{-t} \) and \( te^{-t} \) are solutions to the homogeneous equation. So, instead we expect a particular solution of the form

\[
\psi(t) = v(t)e^{-t}
\]

\[
\psi'(t) = v'e^{-t} - ve^{-t}
\]

\[
\psi''(t) = v''e^{-t} - 2v'e^{-t} + ve^{-t}
\]

where \( v \) is a polynomial. Let’s plug this in to get a better feeling for what degree \( v \) needs to be.

\[
\begin{align*}
\psi'' & = e^{-t}(v'' - 2v' + v) \\
+2\psi' & = e^{-t}(2v' - 2v) \\
-\psi & = e^{-t}(v)
\end{align*}
\]

\[
e^{-t}\left(1\right) = e^{-t}\left(v'\right).
\]

Notice that if the degree of \( v \) is \( m \), then the degree of \( v'' \) is \( m - 2 \). Because we need this to be a constant polynomial, we need \( m - 2 = 0 \), i.e., \( m = 2 \). Therefore,

\[
\psi = v(t)e^{-t} = e^{-t}(a + bt + ct^2).
\]

Notice that since \( e^{-t} \) and \( te^{-t} \) are solutions to the homogeneous equation, we have

\[
L(\psi) = aL(e^{-t}) + bL(te^{-t}) + cL(t^2e^{-t}) = 0 + 0 + cL(t^2e^{-t}).
\]

Thus, our choices for \( a \) and \( b \) are irrelevant, as they cannot contribute towards a nonzero righthand side. Thus, we can set \( a = b = 0 \) right away. If this is the case, and if we plug in this refined guess, we get

\[
e^{-t}(1) = e^t\left(2c\right)
\]

By equating coefficients, we have

\[
c = \frac{1}{2}.
\]
and so
\[ \psi = \frac{1}{2} t^2 e^{-t}. \]

Alternatively, notice that in our work above,
\[ e^{-t} = e^{-t} v''. \]

Therefore,
\[ v'' = 1 \Rightarrow v' = t + c_1 \Rightarrow v = \frac{1}{2} t^2 + c_1 t + c_2, \]

where \( c_1 \) and \( c_2 \) are arbitrary. We set \( c_1 = c_2 = 0 \) to obtain
\[ \psi = \frac{1}{2} t^2 e^{-t}. \]

7. \( y'' + 4y = t \sin(2t) \)

**Solution:** For the moment, let us work on the related problem
\[ y'' + 4y = te^{2it}. \]

Then the righthand side is an exponential times a polynomial, so we expect \( \Psi \) of the form
\[ \Psi = v(t) e^{2it}. \]

\[
\begin{align*}
\Psi' &= v'(t) e^{2it} + 2iv(t) e^{2it} \\
\Psi'' &= v''(t) e^{2it} + 4iv'(t) e^{2it} - 4v(t) e^{2it},
\end{align*}
\]

where \( v \) is a polynomial. Now, if we plug this guess into the (modified) DE, we have
\[
\begin{align*}
\Psi'' + 4\Psi &= e^{2it} v'' + 4iv' - 4v \\
\frac{+4\Psi}{e^{2it}(t)} &= e^{2it} (v'' + 4iv') - 4v.
\end{align*}
\]

Now, if \( v \) is a polynomial of degree \( m \), then \( v'' + 4iv' \) is of degree \( m - 1 \). Therefore, as the polynomial of the RHS is of degree 0, \( m - 1 = 1 \), i.e., \( m = 2 \). Therefore, \( \Psi \) is of the form
\[ \Psi = (a + bt + ct^2) e^{2it}, \]

and if we plug this refined guess into our DE, we have
\[ e^{2it} = e^{2it} \left( t(8ci) + 1(2c + 4bi) \right), \]

and so
\[ c = -\frac{1}{8} i, b = \frac{1}{16}. \]

Since there were no conditions on \( a \), it is a free parameter. So, we set \( a = 0 \). Thus,
\[ \Psi = \frac{1}{16} t (1 - 2it) e^{2it}. \]

Next, we find that
\[
\begin{align*}
\Psi &= \frac{1}{16} t (1 - 2it) e^{2it} \\
&= \frac{1}{16} t \left( \left[ \cos(2t) + i \sin(2t) \right] - 2it \left[ \cos(2t) + i \sin(2t) \right] \right) \\
&= \frac{1}{16} t \left( \cos(2t) + i \sin(2t) \right) - \frac{1}{16} t \left( 2t \cos(2t) + 2i \sin(2t) \right) \\
&= \frac{1}{16} t \left( \cos(2t) + 2t \sin(2t) + i \left[ -2t \cos(2t) + \sin(2t) \right] \right).
\end{align*}
\]
Now, if \( L(y) = y'' + 4y \) is the linear operator for the LHS, then
\[
L(\Psi) = te^{2it} = t\cos(2t) + it\sin(2t).
\]

Therefore, by equating imaginary and real parts (and by using rules of linear operators),
\[
L \left( \frac{1}{16} \left(t \cos 2t + 2t^2 \sin(2t) \right) \right) = t \cos(2t),
\]
and
\[
L \left( \frac{1}{16} \left(t \sin(2t) - 2t^2 \cos(2t) \right) \right) = t \sin(2t).
\]

Because our original RHS was \( t \sin(2t) \), we choose
\[
\psi(t) = \frac{1}{16} \left(t \sin(2t) - 2t^2 \cos(2t) \right).
\]

9. \( y'' - 2y' + 5y = 2\cos^2 t \)

**Solution:** First, recall the half-angle identity:
\[
\cos^2 t = \frac{1 + \cos(2t)}{2}.
\]

Therefore, the DE is equivalent to
\[
L(y) = y'' - 2y' + 5y = 1 + \cos(2t).
\]

Let us consider the related DE
\[
L(y) = 1 + e^{2it}.
\]

Notice first that if \( y = \frac{1}{5} \), then
\[
y'' - 2y' + 5y = \frac{1}{5} 5 = 1.
\]

So, if we can find a \( \Psi \) such that
\[
L(\Psi) = e^{2it},
\]
then
\[
L \left( \frac{1}{5} + \Psi \right) = 1 + e^{2it}.
\]

Now, we assume that \( \Psi \) is of the form
\[
\Psi = v(t)e^{2it},
\]
\[
\Psi' = v'e^{2it} + 2ive^{2it},
\]
\[
\Psi'' = v''e^{2it} + 4iv'e^{2it} - 4ve^{2it},
\]

where \( v \) is a polynomial. Now, if we plug this guess into the (modified) DE, we have
\[
\Psi'' e^{2it}(v'' + 4iv' - 4) - 2\Psi' e^{2it}(-2v' - 4iv) + 5\Psi e^{2it}(5v) = e^{2it}(v'' + v'(4i - 2) + v(1 - 4i)).
\]

Now, we see that if \( v \) is a \( m \)-th degree polynomial, then \( v'' + v'(4i - 2) + v(1 - 4i) \) is also \( m \)-th degree. We need this to be a zero-th degree polynomial, i.e., \( m = 0 \), and so \( v \) is of the form
\[
v = a,
\]

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whereby
\[ 0 + 0 + a(1 - 4i) = 1 \Rightarrow a = \frac{1}{1 - 4i} = \frac{1 + 4i}{1 + 16} = \frac{1 + 4i}{17}. \]

Thus,
\[ \Psi = \frac{1 + 4i}{17} e^{2it} \]
\[ = \frac{1}{17} \left( \cos(2t) - 4 \sin(2t) \right) + i \left( \sin(2t) + 4 \cos(2t) \right). \]

Thus,
\[ L\left(\frac{1}{5} + \Psi\right) = 1 + e^{2it} = 1 + \cos(2t) + i \sin(2t). \]

Therefore, by equating real parts (and using linear operator theory)
\[ L\left(\frac{1}{5} + \text{re} \left(\Psi(t)\right)\right) = 1 + \cos(2t), \]
and so
\[ \psi(t) = \frac{1}{5} \left( 1 + \frac{1}{17} \left( \cos(2t) - 4 \sin(2t) \right) \right) \]
does the trick. (i.e., \( L(\psi) = 1 + \cos(2t) = 2 \cos^2(2t). \))

10. \( y'' - 2y' + 5y = 2(\cos^2 t)e^t \)

**Solution:** First, notice that by the half-angle identity,
\[ 2 \cos^2 t e^t = 2 \left( \frac{1 + \cos(2t)}{2} \right) e^t = e^t + e^t \cos(2t). \]

So, let us study two related problems:
\[ L(y) = y'' - 2y' + 5y = e^t, \quad (2) \]
and
\[ L(y) = y'' - 2y' + 5y = e^t e^{2it} = e^{(1+2i)t} = e^{\alpha t}, \quad (3) \]
where \( \alpha = 1 + 2i. \)

Suppose we have \( \psi_1 \) that satisfies (2) and \( \psi_2 \) that satisfies (3). Then
\[ L(\psi_1 + \psi_2) = L(\psi_1) + L(\psi_2) = e^t + e^{\alpha t} = e^t \left( 1 + e^{2it} \right) = e^t \left( 1 + \cos(2t) + i \sin(2t) \right). \]

If so, then the real part of \( \psi_1 + \psi_2 \) satisfies the real part of the righthand side, namely \( e^t \left( 1 + \cos(2t) \right) = 2 \cos^2(2t)e^t. \)

Let us find such a \( \psi_1 \). Assume that \( \psi_1 \) is of the form
\[ \psi_1 = e^t p(t) \]
\[ \psi'_1 = e^t p' + e^t p \]
\[ \psi''_1 = e^t p'' + e^t 2p' + e^t p, \]
where \( p \) is a polynomial of degree \( m \). Then
\[
\begin{align*}
\psi''_1 &= e^t (p'' + 2p' + p) \\
-2\psi'_1 &= e^t (-2p' - 2p) \\
+5\psi_1 &= e^t (5p) \\
e^t(1) &= e^t (p'' + 4p)
\end{align*}
\]
Now, for the equality to hold, we must have $1 = p'' + 4p$. The left side of this equation is a polynomial of degree 0, and the right is a polynomial of degree $m$. Therefore, $m = 0$, and $p$ is a constant. Hence, 

$$1 = 4p \Rightarrow p = \frac{1}{4},$$

and so 

$$\psi_1 = \frac{1}{4}e^t.$$

Now, let’s look for a $\psi_2$, which we assume is of the form 

$$\psi_2 = e^{\alpha t}q(t)$$

$$\psi''_2 = e^{\alpha t}(q'' + 2\alpha q' + q)$$

$$-2\psi'_2 = e^{\alpha t}(-2q'' - 2\alpha q)$$

$$+5\psi_2 = e^{\alpha t}(+5q)$$

$$e^{\alpha t}(1) = e^{\alpha t}\left(q'' + 2\alpha q' + q\right).$$

Before we proceed, let’s see if we can simplify this equation. Notice that the coefficient for $q$ is the characteristic polynomial for the homogeneous equation, evaluated at $\alpha$. Hmmmmm... :-) This suggests that we take a peek at the characteristic equation and find its roots:

$$r^2 - 2r + 5 = 0 \Rightarrow r = 1 \pm 2i.$$

Notice, then, that $\alpha = 1 + 2i$ is a root of the characteristic equation, and so $\alpha^2 - 2\alpha + 5 = 0$. Thus, the equation above simplifies to 

$$e^{\alpha t}(1) = e^{\alpha t}\left(q'' + 4iq'\right),$$

since $\alpha - 1 = 2i$. Thus, 

$$1 = q'' + 4iq'.$$

The left side is a polynomial of degree 0, and the right side is a polynomial of degree $n - 1$. Therefore, $n = 1$, so $q(t) = a + bt$. Hence, 

$$1 = (a + bt)'' + 4i(a + bt)' = 4ib \Rightarrow b = \frac{1}{4i} = -\frac{1}{4}i.$$

Since $a$ is underdetermined, we set $a = 0$. Thus, 

$$\psi_2 = -\frac{1}{4}ite^{\alpha t} = \frac{1}{4}te^t\left(-i\cos(2t) - i^2 \sin(2t)\right) = \frac{1}{4}te^t\sin(2t) - i\frac{1}{4}te^t\cos(2t).$$

Therefore, the real part of $\psi_1 + \psi_2$ is 

$$\psi = \frac{1}{4}te^t + \frac{1}{4}te^t\sin(2t) = \frac{1}{4}e^t\left(1 + t\sin(2t)\right).$$

You should indeed check to see that this satisfies the original differential equation as a particular solution. (I did in Maple. It works.)

11. $y'' + y' - 6y = \sin t + te^{2t}$

**Hint:** Consider the two problems:

$$L(y) = e^{it}$$

and 

$$L(y) = te^{2t}.$$ 

Find a $\psi_1$ that satisfies the first, and a $\psi_2$ that satisfies the second. Take the imaginary part of $\psi_1$ and add it to $\psi_2$. 

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13. $y'' - 3y' + 2y = e^t + e^{2t}$

**Hint:** I’d do these two problems separately:

$L(y) = e^t$

and

$L(y) = e^{2t}$.

Find $\psi_1$ for the first and $\psi_2$ for the second, and add the results.

15. $y'' + y = \cos t \cos 2t$

**Hint:** This one is trickier. Consider the following:

\[
\cos(t) \cos(2t) = \frac{(e^{it} + e^{-it})}{2} \frac{(e^{2it} + e^{-2it})}{2}
= \frac{1}{4} (e^{3it} + e^{it} + e^{-it} + e^{-3it})
= \frac{1}{4} (2 \cos(3t) + 2 \cos(t))
= \frac{1}{2} \cos(3t) + \frac{1}{2} \cos(t).
\]

So, I’d consider two related problems:

$L(y) = \frac{1}{2} e^{3it}$

and

$L(y) = \frac{1}{2} e^{it}$.

Find $\psi_1$ and $\psi_2$ that solve each, respectively. Add their results, and take the real part.