Page 136, # 4 (5 points): Let \( L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t) \), and suppose that \( L[t^2] = t + 1 \) and \( L[t] = 2t + 2 \). Show that \( y(t) = t - 2t^2 \) is a solution of \( y'' + p(t)y' + q(t)y = 0 \).

Solution: First, note that \( L \) is a linear operator. (You should be able to prove this on your own.) Therefore,

\[
L[y] = L[t - 2t^2] = L[t] - 2L[t^2] = (2t + 2) - 2(t + 1) = 0.
\]

Therefore,

\[
L[y] = y'' + py' + qy = 0,
\]

i.e., \( y \) is a solution to the differential equation.

Page 140, # 4 (2 points): Find the general solution of \( 3y'' + 6y' + 2y = 0 \).

Solution: The characteristic equation is \( 3r^2 + 6r + 2 = 0 \), whose solutions are \(-1 - \frac{\sqrt{3}}{3}\) and \(-1 + \frac{\sqrt{3}}{3}\). As these are real and distinct roots, the general solution is

\[
y(t) = c e^{(-1-\frac{\sqrt{3}}{3})t} + d e^{(-1+\frac{\sqrt{3}}{3})t}.
\]

Page 144, # 2 (3 points): Find the general solution of \( 2y'' + 3y' + 4y = 0 \).

Solution: The characteristic equation is \( 2r^2 + 3r + 4 = 0 \). Its solutions are

\[
r_1 = \alpha - i\beta \quad r_2 = \alpha + i\beta
\]

where

\[
\alpha = -\frac{3}{4}, \quad \beta = \frac{\sqrt{23}}{4}.
\]

If we exponentiate \( r_2t \), we obtain a complex-valued solution

\[
e^{r_2t} = e^{\alpha t} e^{i\beta t} = e^{\alpha t} e^{i\beta t} = e^{\alpha t} \cos(\beta t) + ie^{\alpha t} \sin(\beta t).
\]

Recall that the real and imaginary parts are both solutions, and they are independent. Therefore, the general solution is

\[
y(t) = c e^{\alpha t} \cos(\beta t) + d e^{\alpha t} \sin(\beta t),
\]

where

\[
\alpha = -\frac{3}{4}
\]

and

\[
\beta = \frac{\sqrt{23}}{4}.
\]

Page 149, # 6 (5 points): Solve the following initial-value problem:

\[
y'' + 2y' + y = 0, \quad y(2) = 1, \quad y'(2) = -1.
\]

Solution: The characteristic equation is \( r^2 + 2r + 1 = (r + 1)^2 = 0 \). There is only one distinct root to this equation: \( r_1 = -1 \). Therefore, the general solution is

\[
y(t) = c e^{-t} + d te^{-t} = (c + dt)e^{-t}.
\]
From the first initial condition, we have

\[ y(2) = (c + 2d)e^{-2} = 1 \Rightarrow c + 2d = e^2, \]

and from the second, we have

\[ y'(2) = de^{-2} + (c + 2d)(-e^{-2}) = -e^{-2}(c + d) = -1 \Rightarrow c + d = e^2. \]

Subtracting the second equation from the first, we have

\[ d = 0, \]

and so \( c = e^2 \), and the solution is

\[ y(t) = e^2e^{-t} = e^{2-t}. \]