Page 197, #8 (10 points): Solve the following initial-value problem:

\[ y'' + (t^2 + 2t + 1)y' - (4 + 4t)y = 0; \quad y(-1) = 0, y'(-1) = 1 \]

Solution: Notice that the initial condition is given at \( t = -1 \). Thus, we seek a series solution of the form

\[
y(t) = \sum_{n \geq 0} a_n (t + 1)^n \]
\[
y'(t) = \sum_{n \geq 1} a_n n (t + 1)^{n-1} \]
\[
y''(t) = \sum_{n \geq 2} a_n n(n-1)(t + 1)^{n-2}. \]

Now, let us rewrite the DE in terms of powers of \( (t + 1) \):

\[
y'' + (t + 1)^2 y' - 4(t + 1)y = 0. \]

Then we compute

\[
(t + 1)^2 y' = \sum_{n \geq 1} a_n n(t + 1)^{n-1} (t + 1)^2 = \sum_{n \geq 1} a_n n(t + 1)^{n+1},
\]

and

\[
-4(t + 1)y = \sum_{n \geq 0} -4a_n (t + 1)^n(t + 1) = \sum_{n \geq 0} -4a_n (t + 1)^{n+1}.
\]

Two of these sums are written in terms of \( (t + 1)^{n+1} \), and so we’ll write the remaining sum in this form. Let \( n - 2 = m + 1 \). Then \( n = m + 3 \), and since \( m + 3 = n \geq 2, m \geq -1 \). Thus, the sum becomes

\[
y''(t) = \sum_{m \geq -1} a_{m+3}(m + 3)(m + 2)(t + 1)^{m+1} = \sum_{n \geq -1} a_{n+3}(n + 3)(n + 2)(t + 1)^{n+1}.
\]

Notice that these sums start at different places. They all have the terms for \( n \geq 1 \) in common, so we’ll split off any additional terms and combine the common parts:

\[
y'' = \left( a_2(2)(1)(t + 1)^0 + a_3(3)(2)(t + 1)^1 + \sum_{n \geq 1} a_{n+3}(n + 3)(n + 2)(t + 1)^{n+1} \right)
\]
\[
+ (t + 1)^2 y' + \left( \sum_{n \geq 1} a_n n(t + 1)^{n+1} \right)
\]
\[
-4(t + 1)y + \left( -4a_0(t + 1)^1 + \sum_{n \geq 1} -4a_n(t + 1)^{n+1} \right)
\]
\[
0 = 2a_2 + \left( 6a_3 - 4a_0 \right)(t + 1)^1 + \sum_{n \geq 1} \left[ (n + 3)(n + 2)a_{n+3} + (n - 4)a_n \right](t + 1)^{n+1}.
\]
Therefore,
\[ a_2 = 0, \quad a_3 = \frac{2}{3}a_0, \]
and
\[ a_{n+3} = \frac{4-n}{(n+3)(n+2)}a_n, \quad n \geq 1. \]
If we let \( n + 3 = m \), then \( n = m - 3 \), and \( m - 3 = n \geq 1 \Rightarrow m \geq 4 \). If so, then the recurrence relation becomes
\[ a_m = \frac{7-m}{m(m-1)}a_{m-3}, \quad m \geq 4. \]
Notice that this makes sense: \( a_0 \) and \( a_1 \) are undetermined, \( a_2 \) and \( a_3 \) were given separately above, and this recurrence relation gives everything else.

Now, since
\[ y = \sum_{n \geq 0} a_n(t+1)^n, \]
we have \( 0 = y(-1) = a_0 \), and \( 1 = y'(-1) = a_1 \). Thus,
\[ 0 = a_0 = a_3 = a_6 = a_9 = \cdots = a_{3\ell}, \quad \ell \geq 0, \]
and
\[ 0 = a_2 = a_5 = a_8 = a_{11} = \cdots = a_{3\ell+2}, \quad \ell \geq 0. \]
Let’s write out the first several remaining terms and see if we can find a pattern.

\[
\begin{array}{c|c}
\ell & a_{3\ell+1} \\
\hline
0 & a_1 = 1 \\
1 & a_4 = \frac{7-1}{(4)(3)}a_1 = \frac{1}{3} \\
2 & a_7 = \frac{7-2}{(17)(6)}a_4 = 0 \\
3 & a_{10} = \frac{10-3}{(10)(9)}a_7 = 0 \\
\vdots & \vdots \\
\end{array}
\]

So, the solution is
\[ y(t) = a_1(t+1) + a_4(t+1)^4 = (t+1) + \frac{1}{4}(t+1)^4. \]
Notice that it’s very easy to check that \( y \) satisfies the initial conditions and the differential equation. You should try that.

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<td>Correctly manipulate series</td>
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<td>Find recurrence relation</td>
<td>3 points</td>
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<td>Find pattern in coefficients</td>
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<tr>
<td>Do initial conditions</td>
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Solve the initial-value problem
\[ t^2 y'' - 3ty' + 4y = 0; \quad y(1) = 1, y'(1) = 0 \]
on the interval \( 0 < t < \infty \).

**Solution:** Okay, this is the Euler equation, so we seek solutions of the form \( t^r \). If we plug this into the DE, we get (after algebraic simplification)
\[ t^r (r^2 - 4r + 4) = t^r (r - 2)^2 = 0. \]
Therefore \( r = 2 \) is a double root. Hence, two independent solutions are
\[ y_1(t) = t^2, \quad y_2(t) = t^2 \ln t, \]
and the general solution is
\[ y(t) = (\alpha + \beta \ln(t))t^2. \]
Now,
\[ 1 = y(1) = (\alpha + \beta(0))(1)^2 = \alpha \Rightarrow \alpha = 1, \]
and
\[ 0 = y'(1) = 2(1) + \beta(1) + 2\beta(1) \ln(1) = 2 + \beta \Rightarrow \beta = -2. \]
Hence,
\[ y(t) = (1 - 2 \ln(t))t^2. \]

**Scoring**

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Determine whether the specified value of \( t \) is a regular singular point of the given differential equation:
\[ (e^t - 1)y'' + e^t y' + y = 0; \quad t = 0. \]

**Solution:** Let us rewrite the equation as
\[ y'' + p(t)y' + q(t)y = 0, \]
where
\[ p(t) = \frac{e^t}{e^t - 1} \]
and
\[ q(t) = \frac{1}{e^t - 1}. \]
Then notice first that \( e^0 - 1 = 0 \), and so \( t = 0 \) is indeed a singular point. It is a regular singular point iff \( tp(t) \) and \( t^2 q(t) \) are analytic at \( t = 0 \). Here’s one way to do that for \( tp(t) = \frac{te^t}{e^t - 1} \).

First, compute the Taylor series for the numerator and the denominator centered at \( t_0 = 0 \):
\[ te^t = t \left( 1 + t + \frac{1}{2} t^2 + \frac{1}{6} t^3 + \cdots \right) = t + \frac{1}{2} t^2 + \frac{1}{6} t^3 + \cdots, \]
and
\[ e^t - 1 = \left( 1 + t + \frac{1}{2} t^2 + \frac{1}{6} t^3 + \cdots \right) - 1 = t + \frac{1}{2} t^2 + \frac{1}{6} t^3 + \cdots. \]
Next, we take their ratio and factor out common powers of \((t - t_0) = t:\)

\[
\begin{align*}
    tp(t) &= \frac{te^t}{e^t - 1} \\
    &= \frac{t + t^2 + \frac{1}{2}t^3 + \frac{1}{6}t^4 + \cdots}{t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots} \\
    &= \frac{1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots}{1 + \frac{1}{2}t + \frac{1}{6}t^2 + \cdots}.
\end{align*}
\]

Notice that if we evaluate at \(t = t_0 = 0\), this ratio of finite, i.e., \(\lim_{t \to 0} tp(t) = 1 < \infty\). Therefore, \(tp(t)\) is analytic at \(t = 0\). (The theory behind this is that because \(te^t\) and \(e^t - 1\) are both analytic at \(t = 0\), and because \(\lim_{t \to 0} \frac{te^t}{e^t - 1}\) is finite, the singularity at \(t = 0\) can be “removed” by using the limit. This fact is known as the Riemann Removable Singularity Theorem in complex analysis.)

Let us take a different approach for \(t^2q(t)\). We compute

\[
\lim_{t \to 0} t^2 = 0
\]

and

\[
\lim_{t \to 0} e^t - 1 = 0,
\]

and so l’Hôpital’s rule applies:

\[
\begin{align*}
    \lim_{t \to \infty} t^2q(t) &= \lim_{t \to \infty} \frac{t^2}{e^t - 1} \\
    &= \lim_{t \to \infty} \frac{2t}{e^t - 1} = 0.
\end{align*}
\]

Because this limit is finite, \(\frac{t^2}{e^t - 1}\) is analytic at \(t = 0\). Therefore \(t = 0\) is a regular singular point.

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<tr>
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<tr>
<td>Do the (t^2q(t)) part right</td>
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<tr>
<td>Interpret the result</td>
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