Page 216, #8 (15 points): Find the general solution of the following equation:

\[2ty'' + (1 - 2t)y' - y = 0.\]

Solution: We seek two linearly independent solutions \(y_1\) and \(y_2\). Notice that this DE is singular at \(t = 0\). Therefore, we seek Frobenius series solutions of the form

\[y(t) = \sum_{n \geq 0} a_n t^{n+r},\]

\[y'(t) = \sum_{n \geq 0} a_n (n+r)t^{n+r-1},\]

\[y''(t) = \sum_{n \geq 0} a_n (n+r)(n+r-1)t^{n+r-2}.\]

Now, we notice that

\[2ty'' = \sum_{n \geq 0} 2a_n (n+r)(n+r-1)t^{n+r-1} = \sum_{n \geq -1} 2a_{n+1} (n+r+1)(n+r)t^{n+r},\]

\[y' = \sum_{n \geq 0} a_n (n+r)t^{n+r-1} = \sum_{n \geq -1} a_{n+1} (n+r+1)t^{n+r},\]

\[-2ty' = \sum_{n \geq 0} -2a_n (n+r)t^{n+r},\]

and \(y\) is as written. Therefore, if we combine these terms and simplify, we have

\[2ty'' + (1 - 2t)y' - y = a_0 t^{r-1} (2r(r-1) + r) + \sum_{n \geq 0} [a_{n+1} (n+r+1)(2(n+r)+1) - a_n (2(n+r)+1)] t^{n+r} = 0\]

The first term gives us the indicial equation:

\[2r(r-1) + r = 2r^2 - 2r + r(2r - 1) = 0.\]

Therefore, \(r = 0\) or \(r = \frac{1}{2}\). Also, the recurrence relation is given by

\[a_{n+1} (n+r+1)(2(n+r)+1) - a_n (2(n+r)+1) = 0, \quad n \geq 0\]

\[\Rightarrow a_{n+1} = \frac{(2(n+r)+1)}{(n+r+1)(2(n+r)+1)} a_n, \quad n \geq 0.\]

Therefore,

\[a_n = \frac{(2(n+r-1)+1)}{(n+r)(2(n+r-1)+1)} a_{n-1}, \quad n \geq 1,\]
which simplifies to
\[ a_n = \frac{1}{(n+r)}a_{n-1}, \quad n \geq 1. \]

Let us compute a solution for \( r = 0 \) and \( a_0 = 1 \). By the recurrence relation,
\[ a_n = \frac{1}{n}a_{n-1}, \quad n \geq 1. \]

Quite clearly (if it’s not clear to you, write out a few terms),
\[ a_n = \frac{1}{n!}, \quad n \geq 1, \]
and so if \( y_1 \) is the solution corresponding to \( r = 0 \), then
\[ y_1 = \sum_{n\geq0} \frac{1}{n!}t^n = e^t. \]

Now, let us compute a second solution \( y_2 \) for \( r = \frac{1}{2} \) and \( a_0 = 1 \). The recurrence relation is
\[ a_n = \frac{1}{n + \frac{1}{2}} = \frac{2}{2n+1}, \quad n \geq 1. \]

Therefore,
\[ a_1 = \frac{2}{3}, \quad a_2 = \frac{2^2}{3 \cdot 5}, \quad a_3 = \frac{2^3}{3 \cdot 5 \cdot 7}, \quad a_4 = \frac{2^4}{3 \cdot 5 \cdot 7 \cdot 9}, \ldots \]
and so
\[ y_2(t) = \sum_{n\geq0} \frac{2^n}{3 \cdot 5 \cdots (2n+1)}t^{n+\frac{1}{2}}, \]
and the general solution is
\[ y(t) = c_1y_1(t) + c_2y_2(t). \]

**Scoring**

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<th>part</th>
<th>value</th>
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<tr>
<td>Start with Frobenius series</td>
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<tr>
<td>Differentiate series</td>
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<tr>
<td>Manipulate series</td>
<td>2 points</td>
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<tr>
<td>Find indicial equation</td>
<td>2 points</td>
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<tr>
<td>Find ( r )</td>
<td>1 point</td>
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<tr>
<td>Find recurrence relation</td>
<td>3 points</td>
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<tr>
<td>Find solution for ( r = 0 )</td>
<td>2 points</td>
</tr>
<tr>
<td>Find solution for ( r = \frac{1}{2} )</td>
<td>2 point</td>
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**Page 223, #2 (5 points):** If I see that you gave this problem an honest effort, you’ll get 5 points. Otherwise, 0.