Midterm Examination II  
(Solutions by Paul Macklin)

Print your name: __________________  __________________

Print your ID #: ______________________________

You have 50 minutes to solve the problems. Good luck!
1. Solve the following equation:
   \[ y'' + 5y' + 4y = (1 + t^2)e^{-t}, \quad y(0) = 0, \quad y'(0) = 1. \]

**Solution:** Let’s find the homogeneous solution first. The characteristic equation is
   \[ r^2 + 5r + 4 = (r + 4)(r + 1) = 0, \]
so two linearly independent solutions of the DE are \( y_1(t) = e^{-4t} \) and \( y_2(t) = e^{-t}. \)

Now, we find a particular solution \( \psi. \) Since the RHS is \((1 + t^2)e^{-t},\) we expect that \( \psi \) has the form
   \[ \psi(t) = q(t)e^{-t}t^m, \]
where \( q(t) \) is a polynomial of degree 2 and \( m \geq 0. \) Because \( e^{-t} \) is a solution of the homogeneous equation but \( te^{-t} \) is not, we expect that \( m = 1. \) Therefore,
   \[ \psi(t) = t \cdot q(t)e^{-t} = p(t)e^{-t}, \]
where \( p(t) = t \cdot q(t). \) So,
   \[
   \begin{align*}
   \psi &= p(t)e^{-t} \\
   \psi' &= e^{-t}(p' - p) \\
   \psi'' &= e^{-t}(p'' - 2p' + p)
   \end{align*}
   \]
Thus,
   \[
   \begin{align*}
   \psi'' + 5\psi' + 4\psi &= e^{-t}(p'' - 2p' + p) + 5e^{-t}(p' - p) + 4e^{-t}p = e^{-t}(p'' + 3p') \\
   e^{-t}(1 + t^2) &= e^{-t}(p'' + 3p'),
   \end{align*}
   \]
and so
   \[ 1 + t^2 = p'' + 3p'. \]
Now, since
   \[ p(t) = t \cdot q(t) = t(a + bt + ct^2) = at + bt^2 + ct^3, \]
we have (after differentiating $p$, etc.)

$$1 + t^2 = 1(3a + 2b) + t(6b + 6c) + t^2(9c),$$

and so

$$c = \frac{1}{9}, \quad b = -\frac{1}{9}, \quad a = \frac{11}{27}$$

Thus,

$$\psi(t) = \left(\frac{11}{27}t - \frac{1}{9}t^2 + \frac{1}{9}t^3\right)e^{-t},$$

and so

$$y(t) = c_1e^{-4t} + c_2e^{-t} + \left(\frac{11}{27}t - \frac{1}{9}t^2 + \frac{1}{9}t^3\right)e^{-t}.$$  

By the initial conditions,

$$y(0) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2,$$

and

$$y'(0) = -4c_1 - c_2 + \frac{11}{27} = 1 \Rightarrow 4c_2 - c_2 = \frac{16}{27},$$

whereby

$$c_2 = \frac{14}{81}, \quad c_1 = -\frac{14}{81}.$$  

So, our final answer is

$$y(t) = -\frac{16}{81}e^{-4t} + \frac{16}{81}e^{-t} + \left(\frac{11}{27} - \frac{1}{9}t + \frac{1}{9}t^2\right)te^{-t}.$$  

B. $y'' + 9y = t\sin(3t), \; y(0) = 1, \; y'(0) = 0$.

Solution: Similar.

2. A. A mass of 0.5 Kg stretches a spring by 0.25 m. Letting $c$ Ns/m denote the damping constant, determine the value of $c$ for which the system is critically damped. For that value of $c$ and assuming that the system, initially in equilibrium, is pushed off with an initial velocity of 1 m/s, determine the maximal elongation experienced by the spring. [Use $g = 10 \text{ m/s}^2$.]
Solution: [Note: This is going to be wordier than your solution needs to be. I'm trying to make it as educational as possible.]

First recall that for a spring problem, the general model is

\[ my'' + cy' + ky = F(t). \]

So, we merely need to identify \( m, c, k, F, \) and the initial conditions. By the first sentence, \( m = \frac{1}{2} \), and we can determine \( k \) as follows. At the equilibrium position, the spring is stretched by \( \frac{1}{4} \) m. Note that at equilibrium (without external forces), all forces are at balance and the spring is not moving, so \( y' = 0 \). Also, the only acceleration of the spring is due to gravity, so \( y'' = -g = -10 \). Therefore,

\[
my'' + cy' + ky = \frac{1}{2} \cdot -10 + c \cdot 0 + k \cdot \frac{1}{4} = 0 \Rightarrow 5 = \frac{1}{4}k \Rightarrow k = 20.
\]

So, the model is now

\[
\frac{1}{2} y'' + cy' + 20y = F(t) \Rightarrow y'' + 2cy' + 40y = 2F(t).
\]

Let us determine \( c \) such that the system is critically damped. In mathematics and science, “critically” means “just barely”, so we want to find the value of \( c \) at which the system “just barely” decays exponentially. Well, solutions of the spring system have one of the three following forms:

\[
y(t) = c_1 e^{at} + c_2 e^{bt},
\]

\[
y(t) = c_1 e^{at} + c_2 te^{at},
\]

or

\[
y(t) = e^{at} (c_1 \cos(bt) + c_2 \sin(bt)).
\]

If \( a < 0 \), then the first two solutions decay to zero without oscillating. The first happens when we have two distinct real (negative) eigenvalues, and the second when we have one repeated (negative) eigenvalue. The third happens when we have two complex eigenvalues. The only difference between these is the behavior of the discriminant: the discriminant is positive or zero when the system decays to zero without oscillation, and the discriminant is negative when this behavior stops. So, the “critical point” at which the system is barely damped is when the discriminant is zero.
Now, the characteristic equation is
\[ r^2 + 2cr + 40 = 0, \]
whose solutions are
\[ r = \frac{-2c \pm \sqrt{4c^2 - 4 \cdot 40}}{2} = -c \pm \sqrt{c^2 - 40}. \]

There is a repeated eigenvalue iff the discriminant \( c^2 - 40 \) equals zero. Thus, \( c^2 = 40 \), and so \( c = \sqrt{40} = 2\sqrt{10} \). This is the critical value of \( c \) at which the system is critically damped.

So, our model is
\[ y'' + 4\sqrt{10}y' + 40y = 2F(t). \]

No external forces are mentioned, so \( F(t) = 0 \). Also, \( y(0) = 0 \) and \( y'(0) = 1 \). So, we have the initial-value problem
\[ y'' + 4\sqrt{10}y' + 40y = y'' + 2cy' + 40y = 0, \quad y(0) = 0, \quad y'(0) = 1. \]

Recall that the solutions to the characteristic equation were \(-c \pm 0 = -2\sqrt{10} \). Therefore, the general solution is
\[ y(t) = ae^{-ct} + bte^{-ct} = (a + bt)e^{-ct}. \]

Since \( y(0) = a = 0 \), we have
\[ y(t) = bte^{-ct}. \]

Since \( y'(0) = 0 + b = 1 \), we have
\[ y(t) = te^{-ct} = te^{-2\sqrt{10}t}. \]

To calculate the maximum elongation of the spring, we must maximize \( y(t) \). Thus, we solve
\[ y'(t) = e^{-ct}(1 - ct) = 0 \Rightarrow t = \frac{1}{c}. \]

At this time,
\[ y \left( \frac{1}{c} \right) = \frac{1}{c} e^{-\frac{1}{c}} = \frac{1}{ce} = \frac{1}{2\sqrt{10}e}. \]
B. A mass of 1 Kg stretches a spring by 0.5 m. Letting $c \, \text{Ns/m}$ denote the damping constant, determine the value of $c$ for which the system is critically damped. For that value of $c$ and assuming that the system, initially in equilibrium, is pushed off with at an initial velocity of 2 m/s, determine the maximal elongation experienced by the spring. [Use $g = 10 \, \text{m/s}^2$.]

**Solution:** [Note: We’ll solve this one much more quickly. See version A for a more detailed discourse. I’d highly recommend you do so.]

The general model is

$$my'' + cy' + ky = F(t).$$

By the first sentence, $m = 1$. To determine $k$,

$$mg = ky \Rightarrow 10 = k \cdot \frac{1}{2} \Rightarrow k = 20.$$

Since no external forces are mentioned, the differential equation is

$$y'' + cy' + 20y = 0.$$

Its characteristic equation is

$$r^2 + cr + 20 = 0,$$

whose solutions are given by

$$r = \frac{-c \pm \sqrt{c^2 - 4 \cdot 20}}{2} = -\frac{1}{2}c \pm \frac{1}{2}\sqrt{c^2 - 80}.$$

This system is critically damped if $c^2 - 80 = 0$, so $c = \sqrt{80} = 4\sqrt{5}$. With this value of $c$, the general solution to the differential equation is

$$y(t) = (a + bt)e^{-\frac{1}{2}ct}.$$

By the first initial condition,

$$y(0) = a = 0,$$

and so

$$y(t) = bte^{-\frac{1}{2}ct}.$$
By the second initial condition,

\[ y'(0) = b = 2, \]

and so

\[ y(t) = 2te^{-\frac{1}{2}ct}. \]

To find the maximum elongation, we must solve

\[ y'(t) = 2 \left( 1 - \frac{1}{2}ct \right) e^{-\frac{1}{2}ct} = 0 \Rightarrow t = \frac{2}{c}. \]

At that time,

\[ y \left( \frac{2}{c} \right) = 2 \cdot \frac{2}{c} e^{-\frac{1}{2} \cdot \frac{2}{c}} = \frac{4}{ce} = \frac{1}{\sqrt{5e}}. \]

3. Verify that

A. \( y_1(t) = e^t \)

B. \( y_1(t) = e^{-t} \)

is a solution of the equation

A. \( (t - 1)y'' - ty' + y = 0 \)

B. \( (t - 1)y'' + ty' + y = 0 \)

and compute a second (linearly independent) solution to the equation for \( t > 1 \).

**Solution A:** First notice that

\[
(t - 1)y_1'' - ty_1' + y_1 = (t - 1)(e^t) - t(e^t) + e^t = e^t(t - 1 - t + 1) = 0,
\]

and so \( y_1 \) is a solution to the DE. We wish to find a second linearly independent solution \( y_2 \). We know that since \( y_1 \) is a solution, we can find a linearly independent solution \( y_2 \) of the form

\[ y_2 = f(t)y_1(t) = f(t)e^t. \]

Plugging this into the differential equation, we have

\[
0 = (t - 1)(f'' + 2f' + f)e^t - t(f' + f)e^t + fe^t = e^t \left( (t - 1)f'' + (t - 2)f' \right).
\]
Therefore,
\[(t - 1)f'' + (t - 2)f' = 0.\]
Let \(v(t) = f'(t)\). Then
\[(t - 1)v' + (t - 2)v = 0 \Rightarrow \frac{v'}{v} = -\frac{t - 2}{t - 1} = -1 + \frac{1}{t - 1}\]
Solving this equation by separation of variables (and noting that \(t > 1\)), we have
\[
\ln |v| = c - t + \ln(t - 1),
\]
and so
\[
|v| = e^{c-e^{-t}}(t - 1).
\]
Notice that \(v \equiv 0\) is an equilibrium solution, and so if \(v'(t_0) = 0\), then \(v(t) = 0\) for all time. We don’t want that. So, either \(v > 0\) for all time or \(v < 0\) for all time. We seek any such \(v\), so we might as well choose one such that \(v\) is always positive. Then
\[
v = (t - 1)e^{-t}
\]
is one such solution. Therefore,
\[
f(t) = \int_1^t (s - 1)e^{-s}ds = -te^{-t},
\]
and so
\[
y_2(t) = f(t)e^{-t} = -te^{-t}e^t = -t.
\]
Now here’s the part where you’ll be mildly annoyed. If we had tried a guess where \(y_2(t) = 0\), then we would have had the \(y_2 = a + bt\), and so
\[
0 = (t - 1)(0) - t(b) + (a + bt) = t(-b + b) + a = a.
\]
Thus, \(a = 0\) and \(b\) is arbitrary, i.e., \(y_2(t) = bt\) for any \(b\) works. In particular, we can choose \(b = -1\). So, we could have guessed the solution pretty easily.

**Solution B:** First notice that
\[
(t - 1)y_1'' + ty_1' + y_1 = (t - 1)(e^{-t}) + t(-e^{-t}) + e^{-t} = e^{-t}(t - 1 - t + 1) = 0,
\]
and so $y_1$ is a solution to the DE. We wish to find a second linearly independent solution $y_2$. We know that since $y_1$ is a solution, we can find a linearly independent solution $y_2$ of the form

$$y_2 = f(t)y_1(t) = f(t)e^{-t}.$$  

Plugging this into the differential equation, we have

$$0 = (t-1)\left(f'' - 2f' + f\right)e^{-t} + t(f' - f)e^{-t} + f e^{-t}$$

Therefore,

$$(t-1)f'' - (t-2)f' = 0.$$  

Let $v(t) = f'(t)$. Then

$$(t-1)v' - (t-2)v = 0 \Rightarrow \frac{v'}{v} = \frac{t-2}{t-1} = 1 - \frac{1}{t-1}$$

Solving this equation by separation of variables (and noting that $t > 1$), we have

$$\ln |v| = c + t - \ln(t-1),$$

and so

$$|v| = e^c e^t \frac{1}{t-1}.$$  

Notice that $v \equiv 0$ is an equilibrium solution, and so if $v'(t_0) = 0$, then $v(t) = 0$ for all time. We don’t want that. So, either $v > 0$ for all time or $v < 0$ for all time. We seek any such $v$, so we might as well choose one such that $v$ is always positive. Then

$$v = \frac{1}{t-1} e^t$$

is one such solution. Therefore,

$$f(t) = \int_1^t \frac{1}{s-1} e^s ds,$$

and so

$$y_2(t) = \int_1^t \frac{1}{s-1} e^{s-t} ds.$$
4. Explain how to obtain two linearly independent solutions for the equation

A. \( y'' + ty' - t^2 y = 0 \)

B. \( y'' - ty' + t^2 y = 0 \)

and then determine the recurrence relation for the coefficients of a power series solution about \( t_0 = 0 \). Compute the first six coefficients in the expansion of two linearly independent solutions.

**Solution A:** We look for two solutions of the form

\[ y(t) = \sum_{n\geq 0} a_n t^n. \]

For the first solution \( y_1 \), we stipulate \( y_1(0) = 1 = a_0, \ y'_1(0) = 0 = a_1 \).

For the second solution \( y_2 \), we stipulate \( y_2(0) = 0 = a_0, \ y'_2(0) = 1 = a_1 \).

Now, let’s work this thing. We calculate

\[-t^2y(t) = \sum_{n>0} -a_n t^{n+2} = \sum_{n \geq 2} -a_{n-2} t^n\]

\[ty' = \sum_{n \geq 1} na_n t^n\]

\[y'' = \sum_{n \geq 2} n(n-1)a_n t^{n-2} = \sum_{n \geq 0} (n+2)(n+1)a_{n+2} t^n,\]

and so

\[y'' + ty' - t^2y = 2a_2 + 6a_3 t + a_1 t + \sum_{n \geq 2} \left( (n+2)(n+1)a_{n+2} + na_n - a_{n-2} \right) t^n.\]

Thus,

\[a_2 = 0, \quad a_3 = -\frac{1}{6} a_1,\]

and

\[a_{n+2} = -\frac{n}{(n+1)(n+2)} a_n + \frac{1}{(n+1)(n+2)} a_{n-2}, \quad n \geq 2.\]
This last equation is the recurrence relation, and it can be rewritten
\[ a_n = -\frac{(n - 2)}{(n - 1)(n)} a_{n-2} + \frac{1}{(n - 1)(n)} a_{n-4} \]
\[ = \frac{1}{n(n-1)} \left( -(n - 2)a_{n-2} + a_{n-4} \right), \quad n \geq 4. \]

Let us find the first six terms of \( y_1 \):

<table>
<thead>
<tr>
<th>( n )</th>
<th>( a_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>(-\frac{1}{6} \cdot 0 = 0)</td>
</tr>
<tr>
<td>4</td>
<td>(\frac{1}{12} (-2 \cdot 0 + 1 \cdot 1) = \frac{1}{12})</td>
</tr>
<tr>
<td>5</td>
<td>(\frac{1}{20} (-3 \cdot 0 + 1 \cdot 0) = 0)</td>
</tr>
</tbody>
</table>

So, the first six terms of the first solution are:
\[ y_1(t) \approx 1 + 0t + 0t^2 + 0t^3 + \frac{1}{12} t^4 + 0t^5. \]

Similarly, for \( y_2 \),

<table>
<thead>
<tr>
<th>( n )</th>
<th>( a_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>(-\frac{1}{5} \cdot 1 = -\frac{1}{5})</td>
</tr>
<tr>
<td>4</td>
<td>(\frac{1}{12} (-2 \cdot 0 + 1 \cdot 0) = 0)</td>
</tr>
<tr>
<td>5</td>
<td>(\frac{1}{20} (-3 \cdot -\frac{1}{5} + 1 \cdot 1) = \frac{3}{20})</td>
</tr>
</tbody>
</table>

and so the first six terms of the second solution are:
\[ y_2(t) \approx 0 + t + 0t^2 - \frac{1}{6} t^3 + 0t^4 + \frac{3}{40} t^5. \]

Solution B: Similar.

5. For the following equations, classify \( t_0 = 0 \) into regular point (r), regular singular (rs) or irregular singular point (is). Please circle your
choice and motivate your answer below.

A.

(a) \( t \sin(t)y'' + (e^t - 1)y' + y = 0 \) \( r \) \( rs \) \( is \)

(b) \( t^3 \cos(t)y'' + \sin(t^2)y' - y = 0 \) \( r \) \( rs \) \( is \)

(c) \( (t - 1)^2y'' + y' - 5y = 0 \) \( r \) \( rs \) \( is \)

(d) \( \frac{1}{t}y'' - \frac{1}{t^2}y' + y = 0 \) \( r \) \( rs \) \( is \)

Solution A.a: This is singular at \( t = 0 \). Let us rewrite the DE:

\[
y'' + \frac{e^t - 1}{t \sin t}y' + \frac{1}{t \sin t}y = 0.
\]

Now,

\[
\lim_{t \to 0} tp(t) = \lim_{t \to 0} \frac{t(e^t - 1)}{t \sin t} = \lim_{t \to 0} \frac{t \left( t + \frac{1}{2}t^2 + \cdots \right)}{t \left( t - \frac{1}{3!}t^3 + \cdots \right)} = \lim_{t \to 0} \frac{t^2 \left( 1 + \frac{1}{2}t + \cdots \right)}{t^2 \left( 1 - \frac{1}{3!}t^2 + \cdots \right)} = \lim_{t \to 0} \frac{1 + \frac{1}{2}t + \cdots}{1 - \frac{1}{3!}t^2 + \cdots} = 1,
\]

which is finite.

Similarly,

\[
\lim_{t \to 0} t^2 q(t) = \lim_{t \to 0} \frac{t^2}{t \sin t} = \lim_{t \to 0} \frac{t}{t \sin t} = \lim_{t \to 0} \frac{t}{t - \frac{1}{3!}t^3 + \cdots} = \lim_{t \to 0} \frac{1}{1 - \frac{1}{3!}t^2 + \cdots} = 1,
\]

which is finite. As both these limits are finite, this is a regular singular point.
Solution A.b: This equation is singular at $t = 0$. Let’s write it in the canonical form:

$$y'' + \frac{\sin(t^2)}{t^3 \cos(t)} y'' - \frac{1}{t^3 \cos(t)} y = 0.$$ 

Now,

$$\lim_{t \to 0} t^2 q(t) = \lim_{t \to 0} \frac{-t^2}{t^3 \cos(t)} = \lim_{t \to 0} \frac{-1}{t \cos(t)} = \lim_{t \to 0} \frac{-1}{t (1 - \frac{1}{2} t^2 + \ldots)} = \lim_{t \to 0} \frac{-1}{t - \frac{1}{2} t^3 + \ldots} = -\infty.$$ 

Therefore, $t = 0$ is an irregular singular point.

Solution A.c: Since the coefficient of $y''$ is not zero at $t = 0$, the equation is regular.

Solution A.d: If we write the equation in more standard form, it is

$$y'' - \frac{1}{t} y' + ty = 0,$$

which is singular at $t = 0$ (due to “division by zero” in the $y'$ term).

Now,

$$tp(t) = -1,$$

and

$$t^2 q(t) = t^3,$$

and as these are both finite $t = 0$, this equation has a regular singular point at $t = 0$.

B.

(a) $t(e^t - 1)y'' + \sin(t)y' + y = 0$ \hspace{1cm} r \hspace{1cm} rs \hspace{1cm} is

(b) $t \sin(t^2) y'' + y' - y = 0$ \hspace{1cm} r \hspace{1cm} rs \hspace{1cm} is

(c) $(t + 1)^2 y'' - y' + 5y = 0$ \hspace{1cm} r \hspace{1cm} rs \hspace{1cm} is

(d) $\frac{1}{t^2} y'' - \frac{1}{t} y' + \frac{1}{t^4} y = 0$ \hspace{1cm} r \hspace{1cm} rs \hspace{1cm} is

Solution B: Similar.