SOLUTIONS

1. Find the first four terms of two linearly independent solutions of the following equation:

\[ 2ty'' + (1 + t)y' - 2y = 0. \]

**Solution:** Notice that the DE has a regular singular point at \( t = 0 \). (You should check this.) So, we seek solutions of the form

\[
y = \sum_{n \geq 0} a_n t^{n+r}
\]

\[
y' = \sum_{n \geq 0} a_n (n+r) t^{n+r-1}
\]

\[
y'' = \sum_{n \geq 0} a_n (n+r)(n+r-1) t^{n+r-2}.
\]

So,

\[
2ty'' = \sum_{n \geq 0} 2a_n (n+r)(n+r-1) t^{n+r-1} = \sum_{n \geq -1} 2a_{n+1} (n+r+1)(n+r) t^{n+r}
\]

\[
y' = \sum_{n \geq 0} a_n (n+r) t^{n+r-1} = \sum_{n \geq -1} a_{n+1} (n+r+1) t^{n+r}
\]

\[
ty' = \sum_{n \geq 0} a_n (n+r) t^{n+r}
\]

\[-2y = \sum_{n \geq 0} -2a_n t^{n+r}.
\]

Putting these all together, we have

\[
0 = 2ty'' + y' + ty' - 2y
= a_0 t^r (2r(r-1) + r)
+ \sum_{n \geq 0} \left\{ 2(n+r+1)(n+r) + (n+r+1) \right\} a_{n+1} + \left( (n+r) - 2 \right) a_n \} t^{n+r}.
\]

Therefore,

\[
2r(r-1) + r = r(2r-1) = 0,
\]

and

\[
a_{n+1} = -\frac{n+r-2}{(2n+2r+1)(n+r+1)} a_n, \quad n \geq 0.
\]

The first of these conditions is the indicial equation, and we conclude that \( r = 0 \) or \( r = \frac{1}{2} \). The second is the recurrence relation, which can be rewritten as

\[
a_m = -\frac{m+r-3}{(2m+2r-1)(m+r)} a_{m-1}, \quad m \geq 1.
\]
Let us fix \( r = 0 \). Then the recurrence relation is

\[ a_m = -\frac{m - 3}{(2m - 1)(m)} a_{m-1}, \quad m \geq 1. \]

and the first four coefficients are \( a_0, \)

\[ a_1 = -\frac{-2}{(1)(1)} a_0 = 2a_0, \]

\[ a_2 = -\frac{-1}{(3)(2)} a_1 = \frac{1}{3} a_0, \]

and

\[ a_3 = -\frac{0}{(5)(3)} a_2 = 0. \]

Notice then that \( a_m = 0 \) for all \( m \geq 3 \), and so the solution is

\[ y_1(t) = a_0 t^0 \left( 1 + 2t + \frac{1}{3} t^2 \right). \]

Now, let us fix \( r = \frac{1}{2} \). Then the recurrence relation is

\[ a_m = -\frac{m + \frac{1}{2} - 3}{(2m + 1 - 1) \left( m + \frac{1}{2} \right)} a_{m-1}, \quad m \geq 1, \]

which can be simplified to

\[ a_m = \frac{5 - 2m}{(2m)(2m + 1)} a_{m-1}, \quad m \geq 1. \]

So, the first four coefficients are \( a_0, \)

\[ a_1 = \frac{3}{(2)(3)} a_0 = \frac{1}{2} a_0, \]

\[ a_2 = \frac{1}{(4)(5)} a_1 = \frac{1}{40} a_0, \]

and

\[ a_3 = -\frac{1}{(6)(7)} a_2 = \frac{-1}{1680} a_0. \]

The first four terms of the solution are

\[ y_2(t) = a_0 t^{\frac{1}{2}} \left( 1 + \frac{1}{2} t + \frac{1}{40} t^2 - \frac{1}{1680} t^3 + \cdots \right). \]
2. Suppose you are solving a differential equation

\[ y'' + p(t)y' + q(t)y = 0, \]

which has a regular singular point at 0. If the indicial equation is

\[ F(r) = (r - 1)(r - 3) \]

and the recurrence relation is

\[ a_n = \frac{14}{n(n + r - 2)} a_{n-1}, \quad n \geq 1. \]

Prove that for one of the roots \( r \) of the indicial equation, this differential equation does not have a solution of the form

\[ y(t) = t^r \sum_{n \geq 0} a_n t^n. \]

**Proof:** Notice that the roots of the indicial equation are 1 and 3. Therefore, if the DE has a solution of the form \( t^r \sum_{n \geq 0} a_n t^n \), then \( r = 1 \) or \( r = 3 \). Suppose that \( r = 1 \). Then the recurrence relation is

\[ a_n = \frac{14}{n(n - 1)} a_{n-1}, \quad n \geq 1. \]

Notice that this expression cannot be evaluated for \( n = 1 \). Therefore, the DE does not have a solution of the form

\[ y(t) = t^1 \sum_{n \geq 0} a_n t^n. \]