1. Suppose that you know that for some function $f(t)$, its Laplace transform is $15 \sin s$. Compute the Laplace transform of the following:

(a) (2 points) $tf(t)$

Solution:

$$
\mathcal{L}(t \cdot f(t)) = -F'(s) = -\frac{d}{ds} (15 \sin(s)) = -15 \cos(s).
$$

(b) (2 points) $t^2 f(t)$

Solution:

$$
\mathcal{L} \left( t^2 \cdot f(t) \right) = \mathcal{L} \left( t \cdot tf(t) \right) = -\frac{d}{ds} \mathcal{L} \left( t \cdot f(t) \right) = -\frac{d}{ds} (-15 \cos(s)) = -15 \sin(s).
$$

(c) (2 points) $e^{5t} f(t)$

Solution:

$$
\mathcal{L}(e^{5t} f(t)) = F(s - 5) = 15 \sin(s - 5),
$$

where $F(s)$ denotes the Laplace transform of $f(t)$.

(d) (2 points) $f'(t)$

Solution: If $\mathcal{L}(f(t)) = F(s)$, then

$$
\mathcal{L}(f'(t)) = sF(s) - f(0) = 15s \sin(s) - f(0).
$$

(e) (2 points) $tf'(t)$

Solution: If $\mathcal{L}(f(t)) = F(s)$, then

$$
\mathcal{L} \left( t \cdot f'(t) \right) = -\frac{d}{ds} \mathcal{L}(f'(t)) = -\frac{d}{ds} (15s \sin(s) - f(0)) = -(15 \sin(s) + 15s \cos(s)).
$$
2. (10 points) Solve the following initial value problem by the method of Laplace transforms:
\[ y'' - 7y' + 6y = e^{2t}, \quad y(0) = 0 \quad y'(0) = 3. \]

**Solution:** Let \( Y(s) = \mathcal{L}(y(t)) \). Then after taking the Laplace transform of both sides and using linearity, we have
\[
\left( s^2Y - sy(0) - y'(0) \right) - 7\left( sY - y(0) \right) + 6Y = \frac{1}{s-2},
\]
If we use \( y(0) = 0 \) and \( y'(0) = 3 \) and solve for \( Y \), we find
\[
Y = \frac{3}{(s-1)(s-6)} + \frac{1}{(s-1)(s-2)(s-6)},
\]
and so
\[
y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1} \left( \frac{3}{(s-1)(s-6)} \right) + \mathcal{L}^{-1} \left( \frac{1}{(s-1)(s-2)(s-6)} \right).
\]
Let us compute the inverse transform on the left first. Notice that by partial fractions, we can write
\[
\frac{3}{(s-1)(s-6)} = \frac{-3}{s-1} + \frac{3}{s-6},
\]
and so
\[
\mathcal{L}^{-1} \left( \frac{3}{(s-1)(s-6)} \right) = -\frac{3}{5} e^t + \frac{3}{5} e^{6t}.
\]
We shall treat the other term in more detail. We seek \( A, B, \) and \( C \) such that
\[
\frac{1}{(s-1)(s-2)(s-6)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-6}.
\]
After cross-multiplying, we have
\[
1 = A(s-2)(s-6) + B(s-1)(s-6) + C(s-1)(s-2),
\]
which must hold for all \( s \). Hence, it is true for \( s = 1 \) in particular:
\[
1 = A(-1)(-5) \Rightarrow A = \frac{1}{5}.
\]
If \( s = 2 \), we have
\[
1 = B(1)(-4) \Rightarrow B = -\frac{1}{4}.
\]
Lastly, if \( s = 6 \), we see that
\[
1 = C(5)(4) \Rightarrow C = \frac{1}{20}.
\]
Therefore,

\[ \mathcal{L}^{-1} \left( \frac{1}{(s-1)(s-2)(s-6)} \right) = \frac{1}{5} \mathcal{L}^{-1} \left( \frac{1}{s-1} \right) - \frac{1}{4} \mathcal{L}^{-1} \left( \frac{1}{s-2} \right) + \frac{1}{20} \mathcal{L}^{-1} \left( \frac{1}{s-6} \right) \]

\[ = \frac{1}{5} e^t - \frac{1}{4} e^{2t} + \frac{1}{20} e^{6t}. \]

If we put these together and combine like terms, we have

\[ y(t) = -\frac{2}{5} e^t + \frac{13}{20} e^{6t} - \frac{1}{4} e^{2t}. \]