MATH 120A Prep: Complex Numbers II

1. Calculate the cube of $1+i$ in rectangular and polar form. Convert the result to verify they are equal.

Solution: In rectangular form:

$$\begin{aligned}(1+i)^3 &= (1+i)(1+i)(1+i)\\ &= (1+i)(1+2i-1)\\ &= (1+i)(2i)\\ &= 2i+2i^2\\ &= -2+2i\end{aligned}$$

$1+i$ has polar form $\sqrt{2}e^{\pi i/4}$, so

$$(1+i)^3 = \left(\sqrt{2}e^{\pi i/4}\right)^3 = 2\sqrt{2}e^{3\pi i/4}$$

Converting this back to rectangular form gives

$$2\sqrt{2}\cos(3\pi/4) + 2\sqrt{2}\sin(3\pi/4)i = 2\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) + 2\sqrt{2}\frac{\sqrt{2}}{2}i = -2+2i$$

so both results match.

2. Write out the set of 8th roots of unity and graph them as in the video.

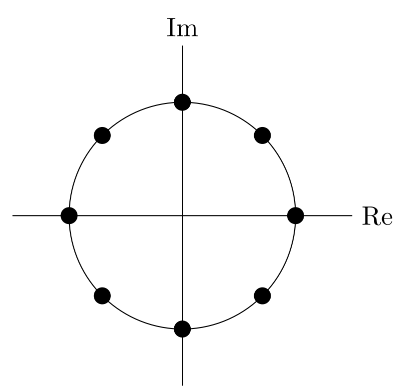
Solution: Recall from the video that the nth roots of unity have the form

$$\{e^{2\pi ik/n} : k = 0, 1, \ldots, n-1\}$$

so for $n = 8$ we have the numbers

$$\{e^{2\pi ik/8} : k = 0, 1, \ldots, 7\} = \{e^{0\pi i/4}, e^{\pi i/4}, e^{2\pi i/4}, \ldots, e^{7\pi i/4}\}$$

The 8th roots of unity are equally spaced around the unit circle with angles of $\pi/4$ between them, so they look like below:



3. Follow the example of finding the nth roots of unity to find all complex solutions to $z^3 = -8$. [Hint: This statement can be rephrased as finding the 3rd roots of -8.]

Solution: We can write -8 in polar form by $-8 = 8e^{\pi i}$, but we can also add/subtract 2π from this angle so all possible forms of -8 are $8e^{(\pi+2\pi k)i}$ for any integer k. To find the third roots of -8 we want the complex numbers $re^{i\theta}$ so that

$$-8 = 8e^{(\pi+2\pi k)i} = \left(re^{i\theta}\right)^3 = r^3e^{3\theta i}$$

To get equality we need $r^3 = 8$ and $3\theta = \pi + 2\pi k$. Thus we have

$$r = 2 \qquad \theta = \frac{\pi}{3} + \frac{2\pi k}{3}$$

Therefore the solutions are

$$\{2e^{(\frac{\pi}{3}+\frac{2\pi k}{3})i} : k \in \mathbb{Z}\}$$

This has repeats since adding 3 to k adds 2π to the angle and leaves the result the same. Therefore we only need $k = 0, 1, 2$ and get the points

$$\{2e^{\frac{\pi i}{3}}, 2e^{\pi i}, 2e^{\frac{5\pi i}{3}}\}$$

Plotting these points looks like below: