# MATH 120A Prep: Equivalence Relations

1. Define a relation on the set of all people by $A \sim B$ if and only if $A$ is at least as tall as $B$. Is this an equivalence relation?

> **Solution:** No. This relation is not symmetric. If person $A$ is strictly taller than person $B$ then $A \sim B$, but $B \not\sim A$.

2. Define a relation on $\mathbb{R}$ by $x \sim y$ if $|x| = |y|$. Show this is an equivalence relation and list the elements of the equivalence classes $[0]$, $[1]$, and $[-3]$.

> **Solution:** Reflexive: Since $x = x$ then $|x| = |x|$ so $x \sim x$.
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> Symmetric: Suppose $x \sim y$, so $|x| = |y|$. Then $|y| = |x|$ and so $y \sim x$.
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> Transitive. Suppose $x \sim y$ and $y \sim z$. Then $|x| = |y|$ and $|y| = |z|$ so $|x| = |z|$ which means $x \sim z$.
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> Since it is Reflexive, Symmetric, and Transitive this relation is an equivalence relation.
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> $[0] = \{0\}$ since no other number has absolute value zero.
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> $[1] = \{1, -1\}$ and $[-3] = \{-3, 3\}$ since only a number and its negative have the same absolute value.

3. Define a relation on $\mathbb{R}^2$ by $(x, y) \sim (u, v)$ if $x^2 + y^2 = u^2 + v^2$. Show this is an equivalence relation and describe the equivalence classes.

> **Solution:** Reflexive: $(x, y) \sim (x, y)$ since $x^2 + y^2 = x^2 + y^2$.
>
> Symmetric: Suppose $(x, y) \sim (u, v)$, so $x^2 + y^2 = u^2 + v^2$. Reversing this equality gives $u^2 + v^2 = x^2 + y^2$ and so $(u, v) \sim (x, y)$.
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> Transitive: Suppose $(x, y) \sim (u, v)$ and $(u, v) \sim (a, b)$. Then $x^2 + y^2 = u^2 + v^2$ and $u^2 + v2 = a^2 + b^2$. Therefore $x^2 + y^2 = a^2 + b^2$ and so $(x, y) \sim (a, b)$.
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> Therefore this is an equivalence relation.
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> Now notice that $x^2 + y^2 = u^2 + v^2$ if and only if $\sqrt{x^2 + y^2} = \sqrt{u^2 + v^2}$ and $\sqrt{x^2 + y^2}$ is the distance from the point $(x, y)$ to the origin $(0, 0)$. So we can rephrase this relation as saying two points are related if and only if they are the same distance from $(0, 0)$. Since the set of points a fixed distance away from at point defines a circle, the equivalence classes are circles in $\mathbb{R}^2$ centered at the origin (with the exception of $[(0, 0)]$ which is just that point).

4. Let $S = \{(a, b) : a, b \in \mathbb{Z}, b \neq 0\}$ Define a relation on $S$ by $(a, b) \sim (c, d)$ if $ad = bc$. It turns out this is actually an equivalence relation. (You can prove this if you like, but it is a bit long and isn't necessary for the problem.) List some elements of the equivalence classes $[(3, 2)]$ and $[(-1, 5)]$. The set of equivalence classes can be represented by a familiar set of numbers, what is it? [Hint: Write $(a, b)$ as $\frac{a}{b}$]

**Solution:** Notice that $(3, 2) \sim (a, b)$ means that $2a = 3b$, so $2|b$ and $3|a$. Write $a = 3k$ and $b = 2l$. Then $6k = 6l$ so we need $k = l$ and so $(a, b) = (3k, 2k)$ for some non-zero integer $k$. (Remember, $b \neq 0$ so $k$ can't be zero either.) In general we can write

$$[(3, 2)] = \{(3k, 2k) : k \neq 0\}$$

but a few elements of this set are $(-3, -2)$, $(6, 4)$, $(-9, -6)$, $(9, 6)$. You may have found other elements but they will look like $(3k, 2k)$.

By the same line of reasoning

$$[(-1, 5)] = \{(-k, 5k) : k \neq 0\}$$

with a few elements including $(1, -5)$, $(-5, 25)$, $(2, -10)$, and so on.

Writing $(a, b)$ in the form $\frac{a}{b}$ makes this look like a fraction or rational number. Indeed, if we think about $(3, 2)$, the elements of its equivalence class look like

$$\left[\frac{3}{2}\right] = \left\{\frac{3k}{2k} : k \neq 0\right\}$$

That is, the fractions that reduce to $3/2$. So the set of equivalence classes from this relation look like the set of rational numbers, where we can cancel terms from the numerator and denominator and get the same value.