Facts to Know:

Function Properties: Consider a function \( f : X \rightarrow Y \).

- **Injective/One-to-one** - Each input goes to a different output.
  
  To prove: Assume \( f(x_1) = f(x_2) \), show \( x_1 = x_2 \).

- **Surjective/Onto** - Every element in the codomain (Y) is the image of an element of X.
  
  To prove: Let \( y \in Y \), want to show there is an \( x \in X \) such that \( f(x) = y \).

- **Bijective** - A function is bijective if and only if it is injective and surjective.

Examples:

1. (a) Determine whether the exponential map \( f : \mathbb{R} \rightarrow \mathbb{R} \), \( f(x) = e^x \) is injective and/or surjective.

   \[ e^x \text{ Injective: Suppose } f(x_1) = f(x_2), \text{ so } e^{x_1} = e^{x_2}. \]

   Apply \( \ln \) to both sides, so \( \ln(e^{x_1}) = \ln(e^{x_2}) \)

   \[ x_1 = x_2. \checkmark \text{ } e^x \text{ is injective.} \]

   \[ \text{Surjective: } e^x \text{ is never negative or zero, so it is not surjective.} \]

   (b) What changes if we consider this as a function \( f : \mathbb{R} \rightarrow \mathbb{R}^+ \) where \( \mathbb{R}^+ = \{ r \in \mathbb{R} : r > 0 \} \)?

   \[ \underline{\text{Injective: Same proof applies.}} \]

   \[ \underline{\text{Surjective: Let } r \in \mathbb{R}^+, \text{ so } r > 0. \text{ Want to find a number } x \text{ in } \mathbb{R} \text{ so } e^x = r. \text{ Let } x = \ln(r).} \]

   \[ e^x = e^{\ln(r)} = r, \text{ so } \underline{\text{surjective}}. \]

   This function is a bijection.
2. Is the map \( g : \mathbb{R}^2 \to \mathbb{R} \) where \( g(x, y) = x^2 + y^2 \) injective? Is it surjective?

Injective: No. \( g(0,0) = 0 \) is not injective since \( 1^2 = 1, (-1)^2 = 1 \).
Choose \( y = 0 \), \( g(x, 0) = x^2 \)
\( g(1,0) = 1^2 - 0^2 = 1 \) \( g(-1,0) = (-1)^2 - 0^2 = 1 \) not injective.

Surjective: \( x^2 \) is positive, \( -y^2 \) is negative.

Case 1: \( r < 0 \)
\( r > 0 \), so we can get \( \sqrt{-r} \)
\( g(0, \sqrt{-r}) = 0^2 - (\sqrt{-r})^2 = -(-r) = r \checkmark \)

Case 2: \( r > 0 \)
Consider \( \sqrt{r} \)
\( g(\sqrt{r}, 0) = \sqrt{r}^2 - 0^2 = r - 0 = r \checkmark \)

\( g(x, y) \) is surjective.

3. Let \( S \) be the set \( \{ (x, y) \in \mathbb{R}^2 : x \neq y \} \). Show the map \( h : S \to \mathbb{R}^2 \) defined by \( h(x, y) = (x - y, x^2 - y^2) \) is injective but not surjective.

Injective: Suppose we have \( (x_1, y_1) \) and \( (x_2, y_2) \) such that \( h(x_1, y_1) = h(x_2, y_2) \). 
\( x_1 - y_1 = x_2 - y_2 \) \( x_1^2 - y_1^2 = x_2^2 - y_2^2 \)
\( x_1^2 - y_1^2 = (x_1 - y_1)(x_1 + y_1) \) \( (x_1, y_1)(x_1 + y_1) = (x_2, y_2)(x_2 + y_2) \)
\( x_1 + y_1 = x_2 + y_2 \)
\( x_1 - y_1 = x_2 - y_2 \)
\( 2x_1 = 2x_2 \) \( x = x_2 \)

Not surjective: \( h(x, 0) = (x, x^2) \)
\( x \neq 0 \) \( x = y \) so not in the domain.