1. Show that $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^2$. Generalize this to a basis for $\mathbb{R}^n$. Conclude that $\mathbb{R}^n$ has dimension $n$. [Note: This is called the standard basis for $\mathbb{R}^n$.]

**Solution:** First we show $\vec{e}_1$ and $\vec{e}_2$ are linearly independent. Suppose $c_1 \vec{e}_1 + c_2 \vec{e}_2 = \vec{0}$, then

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The equality of these vectors means $c_1 = 0$ and $c_2 = 0$ as desired. Now we show they span $\mathbb{R}^2$. Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be any vector in $\mathbb{R}^2$. Then

$$v_1 \vec{e}_1 + v_2 \vec{e}_2 = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{v}$$

as we wanted. Since these vectors are linearly independent and span $\mathbb{R}^2$, they are a basis for $\mathbb{R}^2$. We can generalize this to a basis for $\mathbb{R}^n$ by considering vectors $\vec{e}_1, \ldots, \vec{e}_n$ where $\vec{e}_i$ is a vector with a 1 in the $i$th row and zeros elsewhere.

This is a basis for $\mathbb{R}^n$ by a very similar proof to the $\mathbb{R}^2$ case. Therefore $\mathbb{R}^n$ has dimension $n$.

2. Prove that the vectors $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ do not span $\mathbb{R}^3$.

**Solution:** We need only show that the matrix with these vectors as columns row reduces to one with a row of all zeros.

$$\begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ -1 & 1 & 3 \end{bmatrix} \xrightarrow{R_3 - R_3 + R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{R_3 - R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of all zeros these vectors cannot span $\mathbb{R}^3$. 


3. Show that two vectors are linearly dependent if and only if one is a scalar multiple of the other.

**Solution:** First suppose that vectors \( \vec{v}_1 \) and \( \vec{v}_2 \) are linearly dependent. Then there are constants \( c_1 \) and \( c_2 \) not both zero so that \( c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0} \).

Case 1: \( c_1 \neq 0 \). Then \( \vec{v}_1 = -c_2/c_1 \vec{v}_2 \) is a scalar multiple of \( \vec{v}_2 \).

Case 2: \( c_2 \neq 0 \). Then \( \vec{v}_2 = -c_1/c_2 \vec{v}_1 \) is a scalar multiple of \( \vec{v}_1 \).

In either case one vector is a scalar multiple of the other.

Now assume \( \vec{v}_1 \) and \( \vec{v}_2 \) are vectors and one is a scalar multiple of the other.

Case 1: \( \vec{v}_1 \) is a multiple of \( \vec{v}_2 \). Then there is a real number \( c \) so \( \vec{v}_1 = c \vec{v}_2 \). Then \( \vec{v}_1 - c \vec{v}_2 = \vec{0} \) so they are linearly dependent.

Case 2: \( \vec{v}_2 \) is a multiple of \( \vec{v}_1 \). Then there is a real number \( c \) so that \( \vec{v}_2 = c \vec{v}_1 \). So \( c \vec{v}_1 - \vec{v}_2 = \vec{0} \) and they are linearly dependent.

In either case the vectors are linearly dependent.