MATH 140A Review: proof by contradiction and the contrapositive

1. Show that $\sqrt{5} + \sqrt{7}$ is irrational.

**Solution:**

*Proof.* By way of contradiction, we assume that $\sqrt{5} + \sqrt{7}$ is rational. By definition, we have $\sqrt{5} + \sqrt{7} = p/q$ for some $p, q \in \mathbb{Z}$ where $q \neq 0$. Let us look for a contradiction. We have

$$5 - 7 = (\sqrt{5} + \sqrt{7})(\sqrt{5} - \sqrt{7}) = p/q(\sqrt{5} - \sqrt{7}).$$

Thus,

$$-2q/p = \sqrt{5} - \sqrt{7}.$$

Adding this to $\sqrt{5} + \sqrt{7} = p/q$ gives us:

$$-2q/p + p/q = (\sqrt{5} - \sqrt{7}) + (\sqrt{5} + \sqrt{7}) = 2\sqrt{5}.$$

Thus,

$$(-2q/p + p/q)/2 = \sqrt{5}.$$

Since we know that the rationals are closed under multiplication and addition, then $(-2q/p + p/q)/2 = \sqrt{5}$ is rational. This is a contradiction because we know that $\sqrt{5}$ is irrational. Thus, $\sqrt{5} + \sqrt{7}$ is irrational.

2. Assume that $x$ is an integer. If $x^7 - 3x^5 + 88$ is odd, then $x$ is odd.

**Solution:**

*Proof.* We will show the contrapositive: If $x$ is even, then $x^7 - 3x^5 + 88$ is even.

Assume $x$ is even. Then $x^7$ and $-3x^5$ are both even. Therefore, $x^7 - 3x^5$ is even. Hence, $x^7 - 3x^5 + 88$ an even number. Since the contrapositive statement is true, then the original statement is true.

3. Find the contrapositive of: Let $a_n$ be a sequence of real numbers. If $\sum_{n=0}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$.

**Solution:** The contrapositive is: If $\lim_{n \to \infty} a_n$ does not exist or does not converge to 0, then $\sum_{n=0}^{\infty} a_n$ does not converge.