MATH 2A/5A Prep: Composition of Functions

1. Let $f(x) = \cos(x)$, $g(x) = x^2 + \pi$. Find the functions $f(g(x))$ and $g(f(x))$, then find $f(g(0))$ and $g(f(0))$

Solution: Let $u = g(x) = x^2 + \pi$, then

$$f(g(x)) = f(u) = \cos(u) = \cos(x^2 + \pi).$$

Let $v = f(x) = \cos(x)$, then

$$g(f(x)) = g(v) = v^2 + \pi = (\cos x)^2 + \pi = \cos^2(x) + \pi.$$

So

$$f(g(0)) = \cos(0^2 + \pi) = \cos(\pi) = -1.$$

And

$$g(f(0)) = \cos^2(0) + \pi = 1^2 + \pi = \pi + 1.$$

2. Let $F(x) = e^{x^2}$. Write $F(x)$ as a composition of functions $f(x) = e^x$ and $g(x) = x^2$.

Solution: We draw the diagram

$$x \xrightarrow{g(x)} x^2 \xrightarrow{f(x)} e^{x^2}$$

So $F(x) = f(g(x))$.

3. Let $F(x) = \sin^3(x^2)$. Write $F(x)$ as a composition of three functions.

Solution: Define $f(x) = x^3$, $g(x) = \sin(x)$, $h(x) = x^2$, and draw the diagram:

$$x \xrightarrow{h(x)} x^2 \xrightarrow{g(x)} \sin(x^2) \xrightarrow{f(x)} [\sin(x^2)]^3 = \sin^3(x^2)$$

So $F(x) = f(g(h(x)))$, or equivalently $F(x) = (f \circ g \circ h)(x)$.

4. Let $f(x) = x^2 - x + 4$. Simplify the expression $\dfrac{f(x+1) - f(x-1)}{2}$.

Solution:

$$f(x+1) = (x+1)^2 - (x+1) + 4 = x^2 + 2x + 1 - x - 1 + 4 = x^2 + x + 4.$$

$$f(x-1) = (x-1)^2 - (x-1) + 4 = x^2 - 2x + 1 - x + 1 + 4 = x^2 - 3x + 6.$$

So

$$\begin{aligned} \frac{f(x+1) - f(x-1)}{2} &= \frac{x^2 + x + 4 - (x^2 - 3x + 6)}{2} \\ &= \frac{x^2 + x + 4 - x^2 + 3x - 6}{2} \\ &= \frac{4x - 2}{2} \\ &= 2x - 1 \end{aligned}$$