MATH 3D Prep: Substitution Rule

1. Find a function $f(x)$ such that $f'(x) = \cos(x)e^{\sin(x)}$ and $f(\pi) = 0$.

Solution: Using the substitution $u = \sin(x)$, then $\frac{du}{dx} = \cos(x)$, so $du = \cos(x)dx$, so

$$\int \cos(x)e^{\sin(x)}dx = \int e^u du = e^u + C = e^{\sin(x)} + C.$$

$f(x)$ is the antiderivative of $\cos(x)e^{\sin(x)}$, so

$$f(x) = e^{\sin(x)} + C$$

for some constant C. Plug in $x = \pi$, $f(x) = 0$, we get

$$0 = e^{\sin(\pi)} + C = e^0 + C = 1 + C.$$

Solve for C, we get $C = -1$. So

$$f(x) = e^{\sin(x)} - 1$$

2. Evaluate the indefinite integral $\displaystyle\int \frac{3}{1-2x}dx$.

Solution: We use substitution $u = 1 - 2x$, $du = -2dx$, then

$$\int \frac{3}{1-2x}dx = \int -\frac{3}{2} \cdot \frac{1}{1-2x} \cdot (-2dx) = \int -\frac{3}{2}\frac{du}{u} = -\frac{3}{2}\ln(u) + C = -\frac{3}{2}\ln(1-2x) + C.$$

3. Evaluate the indefinite integral $\displaystyle\int \frac{5}{x^2-2x+5}dx$.

Solution: We first complete a square in the denominator,

$$\int \frac{5}{x^2-2x+5}dx = \int \frac{5}{x^2-2x+1+4}dx = \int \frac{5}{(x-1)^2+4}dx = \int \frac{5}{4}\cdot\frac{1}{\frac{(x-1)^2}{4}+1}dx = \int \frac{5}{4}\cdot\frac{1}{(\frac{x-1}{2})^2+1}dx$$

Then take $u = \frac{x-1}{2}$, $du = \frac{dx}{2}$,

$$\int \frac{5}{4}\cdot\frac{1}{(\frac{x-1}{2})^2+1}dx = \int \frac{5}{2}\cdot\frac{1}{(\frac{x-1}{2})^2+1}\cdot\frac{dx}{2} = \int \frac{5}{2}\cdot\frac{1}{u^2+1}du = \frac{5}{2}\arctan(u)+C = \frac{5}{2}\arctan\left(\frac{x-1}{2}\right)+C$$