MATH 3D Prep: Improper Integration

1. Evaluate $\int_0^\infty e^{5-t}dt$

Solution: Note that
$$\int e^{5-t}dt = -e^{5-t} + C$$
So
$$\begin{aligned}\int_0^\infty e^{5-t}dt &= \lim_{b\to\infty}\int_0^b -e^{5-t}dt = \lim_{b\to\infty}\left[-e^{5-t}\right]_0^b\\ &= \lim_{b\to\infty}\left[-e^{5-b}+e^5\right]\\ &= e^5 + \lim_{b\to\infty}\frac{-e^5}{e^b}\\ &= e^5 + 0 \quad \big(\text{because} - e^5 \text{ is constant and } e^b\to\infty \text{ when } b\to\infty\big)\\ &= e^5\end{aligned}$$

2. Evaluate $\int_0^\infty t^2e^{-t}dt$

Solution: Using integration by part with $u = t^2$, $dv = e^{-t}dt$, we get
$$\int t^2e^{-t}dt = -t^2e^{-t} - \int -2te^{-t}dt = -t^2e^{-t} + 2\int te^{-t}dt.$$
Then by integration part another choice $u = t$, $dv = e^{-t}dt$, we get
$$\int te^{-t}dt = -te^{-t} - \int -e^{-t}dt = -te^{-t} + \int e^{-t}dt = -te^{-t} - e^{-t}.$$
Plug the second equation into the first one, we get
$$\int t^2e^{-t}dt = -t^2e^{-t} + 2\int te^{-t}dt = -t^2e^{-t} + 2\left(-te^{-t} - e^{-t}\right) = e^{-t}(-t^2-2t-2).$$
So
$$\begin{aligned}\int_0^\infty t^2e^{-t}dt &= \lim_{b\to\infty}\int_0^b t^2e^{-t}dt\\ &= \lim_{b\to\infty} e^{-t}(-t^2-2t-2)\Big]_0^b\\ &= \lim_{b\to\infty}\frac{-b^2-2b-2}{e^b} + \frac{0+0+2}{e^0}\\ &= 2 + \lim_{b\to\infty}\frac{-b^2-2b-2}{e^b} \quad \left(\text{the limit is } \frac{\infty}{\infty} - \text{type, can apply l'Hopital's Rule.}\right)\\ &= 2 + \lim_{b\to\infty}\frac{-2b-2}{e^b} \quad (\text{by taking derivative with respect to } b.)\\ &\qquad \left(\text{the limit is again } \frac{\infty}{\infty} - \text{type, can apply l'Hopital's Rule.}\right)\\ &= 2 + \lim_{b\to\infty}\frac{-2}{e^b} \quad (\text{the denominator goes to } \infty \text{ and numerator is constant, so limit is } 0.)\\ &= 2 + 0 = 2\end{aligned}$$