MATH 3D Prep: Integrals Involving Extra Variable

1. Find a function $f(x)$ such that $f'(x) = \sin(x^2)$ and $f(\pi) = 1$

Solution: By Fundamental Theorem of Calculus, if we take

$$f(x) = \int_{\pi}^{x} \sin(t^2)dt + C$$

for some constant C, then we have $f'(x) = \sin(x^2)$. We then use the condition $f(\pi) = 1$ to find C. Plugging in $x = \pi$ and $f(x) = 1$, we get

$$1 = \int_{\pi}^{\pi} \sin(t^2)dt + C = 0 + C$$

So $C = 1$. Therefore

$$f(x) = \int_{\pi}^{x} \sin(t^2)dt + 1$$

2. Use the identity

$$\cos\theta \sin\varphi = \frac{1}{2}[\sin(\theta+\phi) - \sin(\theta-\phi)]$$

to evaluate the integral $\displaystyle\int_0^{\pi} \sin(t)\cos(x-t)dt$.

Solution:

$$\begin{aligned}
\int_0^{\pi} \sin(t)\cos(x-t)dt &= \int_0^{\pi} \frac{1}{2}\left[\sin(t+x-t) - \sin(x-t-t)\right]dt \\
&= \frac{1}{2}\int_0^{\pi} \sin(x) - \sin(x-2t)dt \\
&= \frac{\pi}{2}\sin(x) - \frac{1}{2}\int_0^{\pi} \sin(x-2t)dt \quad \text{(because } x \text{ is regarded as a constant.)} \\
&= \frac{\pi}{2}\sin(x) - \frac{1}{2}\left[\frac{1}{2}\cos(x-2t)\right]_0^{\pi} \quad \text{(use substitution} u = x-2t,\ du = -2dt.) \\
&= \frac{\pi}{2}\sin(x) - \frac{1}{2}[\cos(x) - \cos(x-2\pi)] \\
&= \frac{\pi}{2}\sin(x) \quad \text{(by periodicity of the cosine function, } \cos(x-2\pi) = \cos(x).)
\end{aligned}$$