MATH 3D Prep: Linear Independence

1. Determine whether the vectors $\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 0 \\ 2 \\ 8 \end{bmatrix}$ are linearly independent.

Solution: The augmented matrix for the vector equation $a_1\vec{u} + a_2\vec{v} + u_3\vec{w} = \vec{0}$ is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 1 & 5 & 8 & 0 \end{array}\right].$$

Then we do row reduction,

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 1 & 5 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -6 & 2 & 0 \\ 0 & 3 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 18 & 0 \\ 0 & 3 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 3 & 8 & 0 \\ 0 & 0 & 18 & 0 \end{array}\right]$$

The last matrix is in row echelon form and there's a pivot in each of the first, the second and the third column. This means the vector equation has only the trivial solution $a_1 = a_2 = a_3 = 0$, so the vectors are linearly independent.

2. Determine whether the functions $x^2 + 2x + 1$, $x^2 - 1$, $x + 1$ are linearly independent

Solution:

Method 1:

$x^2 + 2x + 1$, $x^2 - 1$, and $x + 1$ correspond to vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ \vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

respectively. So it suffices to determine whether the three vectors are linearly independent. We row reduce the augmented matrix,

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix is in row echelon form, and the third column has no pivot. So a_3 is free. We take $a_3 = 2$, then using the echelon form, we can get $a_2 = 1$, $a_1 = -a_2 = -1$. So

$$-\vec{u} + \vec{v} + 2\vec{w} = \vec{0}$$

So $\vec{u}, \vec{v}, \vec{w}$ are not linearly independent. Therefore the functions $x^2 + 2x + 1$, $x^2 - 1$, and $x + 1$ are not linearly independent.

Method 2:

Suppose

$$a_1(x^2+2x+1)+a_2(x^2-1)+a_3(x+1)=0,$$

then

$$(a_1+a_2)x^2+(2a_1+a_3)x+(a_1-a_2+a_3)=0=0x^2+0x+0.$$

So

$$\begin{cases} a_1 & +a_2 & & =0 \\ 2a_1 & & +a_3 & =0 \\ a_1 & -a_2 & +a_3 & =0 \end{cases}$$

The augmented matrix is same as the matrix from method 1. So solving the linear system gives nontrivial solution $a_1=-1,\ a_2=1,\ a_3=2$, therefore

$$-(x^2+2x+1)+(x^2-1)+2(x+1)=0.$$

So the functions x^2+2x+1, x^2-1, and $x+1$ are not linearly independent.