MATH 3D Prep: Eigenvalues and Eigenvectors

1. Let $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$, find all eigenvalues, and for each eigenvalue, find a basis for the corresponding eigenspace.

Solution: To find all eigenvalues, we solve

$$\begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0.$$

This is same as

$$\begin{aligned} (3-\lambda)^2(2-\lambda) - (2-\lambda) &= 0 \\ \left[(3-\lambda)^2 - 1\right](2-\lambda) &= 0 \\ (3-\lambda-1)(3-\lambda+1)(2-\lambda) &= 0 \\ -(\lambda-2)^2(\lambda-4) = 0 \end{aligned}$$

So there are 2 solutions, $\lambda = 2$ and $\lambda = 4$.

For $\lambda = 2$:
The matrix equation $(A - 2I)\vec{x} = \vec{0}$ has the augmented matrix

$$(A - 2I|\vec{0}) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix is in row echelon form. There's no pivot in the second and third column, so x_2 and x_3 are free variables, and $x_1 + x_3 = 0$, so $x_1 = -x_3$. Then

$$\vec{x} = \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

So a basis for the eigenspace corresponding to $\lambda = 2$ is the set

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

For $\lambda = 4$:
The matrix equation $(A - 4I)\vec{x} = \vec{0}$ has the augmented matrix

$$(A - 4I|\vec{0}) = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix is in reduced row echelon form. There's no pivot in the third column, so x_3 is free variable, and the second row implies $x_2 = 0$, first row implies $x_1 = x_3$. So

$$\vec{x} = \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

So a basis for the eigenspace corresponding to $\lambda = 4$ is the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$