MATH 3D Prep: Sigma Notations

1. Rewrite the sum $\displaystyle\sum_{i=0}^{m-2} \frac{i+2}{m} \ln(1+i)$ as a sum that ends at $i = m$.

Solution: We want to use a new index j such that $j = m$ when $i = m - 2$. So $j = i + 2$, or equivalently $i = j - 2$. Replace i by $j - 2$ everywhere in the sum, we get

$$\sum_{j-2=0}^{m} \frac{j-2+2}{m} \ln(1+j-2)$$

This is same as

$$\sum_{j=2}^{m} \frac{j}{m} \ln(j-1)$$

Renaming the index j by i, we get the desired answer

$$\sum_{i=2}^{m} \frac{i}{m} \ln(i-1)$$

2. Write the sum $\displaystyle\sum_{n=1}^{\infty} n(n+1)x^{n+1} + \sum_{n=1}^{\infty}(n+3)x^{n-1}$ in the form $\displaystyle\sum_{n=c}^{\infty} a_n x^n$.

Solution: We deal with the two power series separately. For $\displaystyle\sum_{n=1}^{\infty} n(n+1)x^{n+1}$, we want to make x^{n+1} to be in the form x^m, so we take $m = n+1$, then $n = m-1$. Replacing n by $m-1$ everywhere in the first sum, we get

$$\sum_{n=1}^{\infty} n(n+1)x^{n+1} = \sum_{m-1=1}^{\infty} (m-1)(m-1+1)x^{m-1+1} = \sum_{m=2}^{\infty}(m-1)mx^m$$

Then we deal with $\displaystyle\sum_{n=1}^{\infty}(n+3)x^{n-1}$. Similarly we let $m = n-1$, then $n = m+1$, again we replace n by $m+1$ everywhere in the second sum,

$$\sum_{n=1}^{\infty}(n+3)x^{n-1} = \sum_{m+1=1}^{\infty} (m+1+3)x^{m+1-1} = \sum_{m=0}^{\infty}(m+4)x^m$$

So

$$\begin{aligned}
\sum_{n=1}^{\infty} n(n+1)x^{n+1} + \sum_{n=1}^{\infty}(n+3)x^{n-1} &= \sum_{m=2}^{\infty}(m-1)mx^m + \sum_{m=0}^{\infty}(m+4)x^m \\
&= \sum_{m=2}^{\infty}(m-1)mx^m + 4x^0 + 5x^1 + \sum_{m=2}^{\infty}(m+4)x^m \\
&= 4x^0 + 5x^1 + \sum_{m=2}^{\infty} \left[(m-1)mx^m + (m+4)x^m\right]
\end{aligned}$$

$$= 4x^0 + 5x^1 + \sum_{m=2}^{\infty} [m^2 - m + m + 4]x^m$$

$$= 4x^0 + 5x^1 + \sum_{m=2}^{\infty} (m^2 + 4)x^m$$

$$= \sum_{m=0}^{\infty} (m^2 + 4)x^m$$

$$= \sum_{n=0}^{\infty} (n^2 + 4)x^n$$