Facts to Know:
Let \( f : \mathbb{R}^2 \to \mathbb{R} \). we define the gradient \( \nabla f \) of \( f \) at the point \( p \) to be the vector:
\[
\nabla f(p) = \begin{pmatrix} \frac{\partial f}{\partial x}(p) \\ \frac{\partial f}{\partial y}(p) \end{pmatrix}
\]

Geometrical interpretation:
- If \( \nabla f(p) \) is a non-zero vector, then the vector \( \nabla f(p) \) is the direction that the most quickly away from \( p \).
- \( ||\nabla f(p)|| \) is the rate of change in direction \( \nabla f(p) \).

The linearization of \( f \) at the point \( p \) is:
\[
f(x, y) \approx L_f(x, y) = f(p) + \nabla f(p) \cdot (x, y - p)
\]

Examples: Let \( f(x, y) = e^{xy^2} \).

1. What is the gradient of \( f \) at the point \( (1, 2) \)?
\[
\nabla f(x, y) = \begin{pmatrix} ye^{xy^2} \\ 2xye^{xy^2} \end{pmatrix}
\]
\[
\nabla f(1, 2) = \begin{pmatrix} 4e^4 \\ 4e^4 \end{pmatrix}
\]

2. What direction increases the fastest from the point \( (1, 2) \) and what is the rate of change in the same direction?

The direction that increases the fastest from the point \( (1, 2) \) is \( \begin{pmatrix} 4e^4 \\ 4e^4 \end{pmatrix} \).

The rate of change in that direction is
\[
||\begin{pmatrix} 4e^4 \\ 4e^4 \end{pmatrix}|| = \sqrt{(4e^4)^2 + (4e^4)^2} = 4e^4 \cdot \sqrt{2}.
\]
3. What is the linearization of $f$ at the point $(1,2)$?

$$L_f(x,y) = e^y + \left( \frac{\partial f}{\partial x} \right)_y \cdot \left( (y) - (\frac{1}{2}) \right)$$

$$= e^y + \left( \frac{\partial f}{\partial x} \right)_y \cdot \left( \frac{x-1}{y-2} \right)$$

$$= e^y + ye^y(x-1) + ye^y(y-2)$$

$$= ye^y \cdot y + ye^y(y-2)$$

$$A \cdot x + B \cdot y + C$$