MATH 112A Review: Line Integrals, Surface Integrals, Parametrization of Curves

Facts to Know:

(Line integral of a scalar field) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar field and let $\vec{r}(t)$ be a bijective parametrization of a curve $C$ in $\mathbb{R}^n$ with parameter $t \in [a, b]$ such that $\vec{r}(a)$ and $\vec{r}(b)$ are the endpoints of $C$. Then the line integral along $C$ is

$$\int_C f(x_1, \ldots, x_n) ds = \int_a^b f(\vec{r}(t)) ||\vec{r}'(t)|| dt.$$

(Line integral of a vector field) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field and let $\vec{r}(t)$ be a bijective parametrization of a curve $C$ in $\mathbb{R}^n$ with parameter $t \in [a, b]$ such that $\vec{r}(a)$ and $\vec{r}(b)$ are the endpoints of $C$. Then the line integral along $C$ is in the direction of $\vec{r}$ is

$$\int_C F(\vec{r}) \cdot d\vec{r} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) dt.$$

(Surface integral of a scalar field) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field and let $\vec{r}(s, t)$ be a parametrization of a surface $S$ in $\mathbb{R}^3$ with $(s, t)$ vary in some region $T$ in the plain. Then, the surface integral over $S$ is given by

$$\iint_S f(x, y, z) dS = \iint_T f(\vec{r}(s, t)) \sqrt{\left(\frac{\partial}{\partial s} \vec{r}(s, t)\right)^2 + \left(\frac{\partial}{\partial t} \vec{r}(s, t)\right)^2} ds dt.$$

(Surface integral of a vector field) Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a vector field and let $\vec{r}(s, t)$ be a parametrization of a surface $S$ in $\mathbb{R}^3$ with $(s, t)$ vary in some region $T$ in the plain. Then, the surface integral over $S$ is given by

$$\iint_S F \cdot d\vec{S} = \iint_T F(\vec{r}(s, t)) \cdot \left(\frac{\partial}{\partial s} \vec{r}(s, t) \times \frac{\partial}{\partial t} \vec{r}(s, t)\right) ds dt.$$

Examples:

1. Let $F(x, y) = (P(x, y), Q(x, y))$ and let $\vec{r}(t) = (x(t), y(t))$ be the parametrization of $C$. What is the line integral?
2. Let \( F(x, y) = (y, -x) \) and consider the parametrization \( \vec{r}(t) = (\cos t, \sin t) \) for \( t \in [0, 2\pi] \) of the unit circle \( C \) with counterclockwise orientation. Compute

\[
\int_C F(\vec{r}) \cdot \vec{r}'(t)dt
\]