Facts to Know:

(Line integral of a scalar field) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar field and let $\vec{r}(t)$ be a bijective parametrization of a curve $C$ in $\mathbb{R}^n$ with parameter $t \in [a, b]$ such that $\vec{r}(a)$ and $\vec{r}(b)$ are the endpoints of $C$. Then the line integral along $C$ is

$$\int_C f(x_1, \ldots, x_n) \, ds = \int_a^b f(\vec{r}(t)) \left\| \vec{r}'(t) \right\| \, dt.$$  

(Line integral of a vector field) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field and let $\vec{r}(t)$ be a bijective parametrization of a curve $C$ in $\mathbb{R}^n$ with parameter $t \in [a, b]$ such that $\vec{r}(a)$ and $\vec{r}(b)$ are the endpoints of $C$. Then the line integral along $C$ is in the direction of $\vec{r}$ is

$$\int_C F(\vec{r}) \cdot d\vec{r} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) \, dt.$$  

(Surface integral of a scalar field) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field and let $\vec{r}(s, t)$ be a parametrization of a surface $S$ in $\mathbb{R}^3$ with $(s, t)$ vary in some region $T$ in the plain. Then, the surface integral over $S$ is given by

$$\iint_S f(x, y, z) \, dS = \iint_T f(\vec{r}(s, t)) \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\| \, ds \, dt.$$  

(Surface integral of a vector field) Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field and let $\vec{r}(s, t)$ be a parametrization of a surface $S$ in $\mathbb{R}^3$ with $(s, t)$ vary in some region $T$ in the plain. Then, the surface integral over $S$ is given by

$$\iint_S F \cdot d\vec{S} = \iint_T F(\vec{r}(s, t)) \cdot \left( \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) \, ds \, dt.$$  

Examples:

1. Let $F(x, y) = (P(x, y), Q(x, y))$ and let $\vec{r}(t) = (x(t), y(t))$ be the parametrization of $C$. What is the line integral?

$$\int_C F(\vec{r}) \cdot d\vec{r} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$= \int_a^b \left( P(x(t), y(t)), Q(x(t), y(t)) \right) \cdot (x'(t), y'(t)) \, dt.$$
\[
\int_a^b P(x(t), y(t)) x'(t) \, dt + \int_a^b Q(x(t), y(t)) y'(t) \, dt
\]

2. Let \( F(x, y) = (y, -x) \) and consider the parametrization \( \tilde{r}(t) = (\cos t, \sin t) \) for \( t \in [0, 2\pi] \) of the unit circle \( C \) with counterclockwise orientation. Compute

\[
\int_C F(\tilde{r}) \cdot \tilde{r}'(t) \, dt = (*)
\]

\[
\tilde{r}'(t) = (-\sin t, \cos t)
\]

\[
F(\tilde{r}(t)) = (\sin t, -\cos t)
\]

\[
F(\tilde{r}(t)) \cdot \tilde{r}'(t) = -\sin^2 t - \cos^2 t = -1
\]

\[
(*) = \int_0^{2\pi} -1 \, dt = -2\pi
\]