Facts to Know:

(Green’s theorem) Let $C$ be a positively oriented, piecewise smooth, simple closed curve in a plane, and let $R$ be the region bounded by $C$:

If $L(x, y)$ and $M(x, y)$ are real-valued functions that have continuous partial derivatives, then

$$\oint_C (L \, dx + M \, dy) = \iint_R \left( \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) \, dA$$

(The gradient as a vector)

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$\text{div } F = \nabla \cdot F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\text{curl } F = \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

(Stoke’s theorem) Let $S$ be a surface bounded by a curve $C$ and $\vec{F}$ be a vector field. Then

$$\iint_S (\nabla \times F) \cdot d\vec{S} = \oint_C F(\vec{r}) \cdot d\vec{r}$$

Note the direction of the line integral:

(Divergence theorem) Let $E$ be a simple solid region and $S$ be the boundary surface of $E$. Then

$$\iint_S F \cdot d\vec{S} = \iiint_E (\nabla \cdot F) \, dV$$
Examples:

1. Evaluate

\[ \int_C (ydx + xdy), = (\ast) \]

where \( C \) is a circle of radius 1 with positive orientation.

\[
(\ast) = \iint_R \left( \frac{\partial}{\partial x} x - \frac{\partial}{\partial y} y \right) dA = 0.
\]

2. Use the divergence theorem to evaluate

\[ \iiint_B \nabla \cdot F \, dV, \]

where \( F(x, y, z) = (y, -1/2y^2, yz + z) \) and let \( S \) be the surface of a sphere of radius 1.

\[
\iiint_S F \cdot d\vec{S} = \iiint_B (\nabla \cdot F) \, dV = \iiint_B 1 \, dV = \frac{4}{3} \pi.
\]

\[
\nabla \cdot F = 0 - y + y + 1 = 1
\]