THE ROYAL SOCIETY

A note on third-order structure functions in turbulence

By Q. Nie[1] and S. Tanveer[2]

[1] *Institute of Mathematics and its Applications, The University of Minnesota, Minneapolis, MN 55455, USA*

[2] *Mathematics Department, The Ohio State University, Columbus, OH 43210, USA*

Received 2 June 1997; revised 6 March 1998; accepted 21 May 1998

Starting from the Navier–Stokes equation, we rigorously prove that a modified third-order structure function, $\tilde{S}_3(r)$, asymptotically equals $-\frac{4}{3}\epsilon r$ (ϵ is the dissipation rate) in an inertial regime. From this result, we rigorously confirm the Kolmogorov four-fifths law, without the Kolmogorov assumption on isotropy. Our definition of the structure function involves a solid angle averaging over all possible orientations of the displacement vector y, besides space-time-averaging. Direct numerical simulation for a highly symmetric flow for a Taylor Reynolds number of up to 155 shows that the flow remains significantly anisotropic and that, without solid angle averaging, the resulting structure functions approximately satisfy these scaling relations over some range of $r = |y|$ for some orientation of y, but not for another.

Keywords: structure functions; turbulence; Navier–Stokes equation; inertial regime; third order

1. Introduction

The subject of structure functions has been of much interest over the last five decades (and continues to be so), since Kolmogorov introduced it in his seminal paper (Kolmogorov 1941*a*) on turbulence. The subject is too vast to review properly in this short paper (see, for example, Monin & Yaglom 1975).

Kolmogorov defined the nth order longitudinal structure functions to be

$$S_{n,K}(r) = \langle |u(x+y,t) - u(x,t)|^n \cos^n \theta_{\delta u,y} \rangle, \tag{1.1}$$

where $u(x,t)$ is the Eulerian velocity field, at a location x (in $\mathbb{R}^3$) at time t, r is the magnitude of the vector y, n is a positive integer, $\theta_{\delta u,y}$ is the angle between $u(x+y,t) - u(x,t)$ and y and $\langle\cdot\rangle$ denotes an ensemble average. In his seminal paper, Kolmogorov (1941*a*) used essentially statistical arguments to conclude that for a homogeneous isotropic flow (for which $S_{n,K}$ is only a function of r), as Reynolds number $Re \to \infty$,

$$S_{n,K}(r) \sim k_n(\epsilon r)^{\zeta_n}, \quad \text{where } \zeta_n = \tfrac{1}{3}n, \tag{1.2}$$

for $\eta \ll r \ll L$ (called the inertial scale), where L is a characteristic energy-producing length-scale, and η is a viscous cut-off scale, with $\eta/L \to 0$ as the Reynolds number $Re \to \infty$. The ϵ appearing in (1.2) is the dissipation rate, assumed to be finite and non-zero in the limit of $Re \to \infty$. The k_n, appearing in (1.2) are universal constants, by Kolmogorov's original argument. An expression for the viscous cut-off scale η can be derived from the Navier–Stokes equation by assuming *a priori* that (1.2) holds for $n = 2$ (see remark 2.4 below). This gives the viscous cut-off (the so-called Kolmogorov length-scale) to be $\eta = \nu^{3/4}/\epsilon^{1/4}$, where ν is the kinematic viscosity. This viscous cut-off length-scale, estimated by Kolmogorov, is consistent with rigorous mathematical results (Constantin *et al.* 1985; Ruelle 1982; Lieb 1984) on the dimension of the global attractor of the Navier–Stokes dynamics (if there exists such an attractor).

Fifty years after Kolmogorov's seminal paper (Kolmogorov 1941*a*), the scope of validity of (1.2) remains a matter of controversy (see Frisch 1995) because, for $n \neq 3$, there are no proofs or derivations of these results that use the Navier–Stokes equation. There is some work on the Kolmogorov spectrum (Lundgren 1982; Pullin & Saffman 1993; Bhattacharjee *et al.* 1995), associated with $n = 2$ structure function, based on modelling of the Navier–Stokes dynamics by assumed vortex structures. While the model predictions do not appear to be critically sensitive to some of the assumptions, the results are still not independent of the model parameters. There has also been some rigorous (Constantin 1997; Constantin *et al.* 1999) upper bounds related to the enstrophy spectrum in two dimensions and the energy spectrum in three dimensions. Constantin & Fefferman (1994) also established rigorous inequality relations for different order structure functions, $S_n(r)$, defined by

$$S_n(r) = \langle |u(x + y, t) - u(x, t)|^n \rangle, \tag{1.3}$$

where $\langle \cdot \rangle$ in their work involves space-time-averaging. Note that these sets of structure functions are different from those originally defined by Kolmogorov; however, there exist relations between S_n and $S_{n,K}$ for isotropic† homogeneous flow (see Monin & Yaglom 1975) and, therefore, S_n will also satisfy relation (1.2) to the same degree as $S_{n,K}$.

Experimental evidence (Gagne 1993; Sreenivasan & Kailasnath 1993; Nelkin 1994; Benzi *et al.* 1993) appears to suggest that the Kolmogorov relation needs to be corrected, at least for $n = 4$. Experimental inaccuracies appear (Frisch 1995) to create uncertainties in the reported results for $n > 4$. The deviation of ζ_n from the predicted $n/3$ of Kolmogorov is popularly known as the intermittency effect, and has occupied the attention of many researchers in recent years. While there exists phenomenological theory (She & Leveque 1994) that predicts the deviation of ζ_n from $n/3$ (for $n \neq 3$) in (1.2), in good agreement with experiment, the relation of intermittency with Navier–Stokes dynamics remains to be understood. Much of the theoretical work in this direction involves modelling and simplification of the Navier–Stokes dynamics (see, for example, L'vov & Procaccia 1996) with a view to capturing the essential physics behind intermittency. One might expect that a simplification describing the essential physics should not violate any exact relation satisfied by the Navier–Stokes dynamics. This highlights the importance of exact relations. These are also helpful to the experimentalist by providing them with checks for consistency.

Unfortunately, there are not many exact relations known for the Navier–Stokes dynamics. Aside from the inequalities mentioned before, until now, the only exact equality involving structure functions that we have been aware of is that $S_{3,K} =$

† Even without isotropy assumption, x–t integration of equation (3.1) and a volume integration with respect to y over a sphere of radius r leads to $r^3 S_{2,K}(r) = \int_0^r \hat{r}^2 S_2(\hat{r})\,\mathrm{d}\hat{r}$ in the inertial regime. This gives the proportionality relation between S_2 and $S_{2,K}$ known before for isotropic flow (Monin & Yaglom 1975).

$-\frac{4}{5}\epsilon r$ as $Re \to \infty$ for r in the inertial regime. This was obtained by Kolmogorov himself (Kolmogorov 1941*b*) by using the Karman–Howarth equation (von Karman & Howarth 1938), that was, in turn, derived from the Navier–Stokes equation for a statistically stationary homogeneous isotropic flow. This has been referred to in the literature as the Kolmogorov four-fifths law.

In this paper, we present a second equality

$$\epsilon^{-1}\tilde{S}_3(r) = -\tfrac{4}{3}r,$$

in an inertial regime, where $\tilde{S}_3$ is a modified third-order structure function. This result is rigorously derived without making *any* assumptions about the flow structure, though the result follows in a rather straightforward manner from the anisotropic generalization of the Karman–Howarth equation (von Karman & Howarth 1938) attributed to Monin (see Monin & Yaglom 1975, p. 402). However, to the best of our knowledge, the results for $\tilde{S}_3(r)$ do not appear anywhere in the existing literature. We also use this result on $\tilde{S}_3(r)$ to rigorously re-derive the Kolmogorov four-fifths law without the Kolmogorov assumption on flow isotropy.

A direct numerical simulation of Navier–Stokes equations for a highly symmetric periodic flow in a box, originally devised by Kida & Ohkitani (1992), was carried out up to a Taylor Reynolds number Re_λ of about 155. The solid angle averaging over all possible orientations of the displacement vector y present in our definition of the structure functions (along with space-time-averaging) makes a numerical computation of the structure functions prohibitively expensive. Instead, we computed structure functions without solid angle averaging for two independent orientations of y. We expected to get approximately the same results in both directions, as would be consistent with an isotropic flow. Instead, we found that up to $Re_\lambda = 155$, the flow was far from isotropic, even for this highly symmetric flow. Without the solid angle averaging, we found that the computed structure functions for $y = (r/\sqrt{3})(1,1,1)$ displayed the theoretical large Reynolds number inertial regime dependences of $\tilde{S}_3(r)$ and $S_{3,K}(r)$ over some range in r. However, for another orientation of y, namely $y = r(1,0,0)$, we find no such scaling regime, at least up to $Re_\lambda = 155$.

For our purposes, we define

$$\tilde{S}_3(r) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \int \frac{\mathrm{d}\Omega}{4\pi}\int \frac{\mathrm{d}x}{L^3}|u(x+y,t)-u(x,t)|^3\cos[\theta_{y,\delta u}]. \tag{1.4}$$

Here, L is some characteristic energy-producing length-scale, and integration with respect to Ω refers to solid angle integration over the spherical surface $|y| = r$. Also, we redefine Kolmogorov's longitudinal structure functions as

$$S_{n,K}(r) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\int \frac{\mathrm{d}\Omega}{4\pi}\int \frac{\mathrm{d}x}{L^3}|u(x+y,t)-u(x,t)|^n\cos^n[\theta_{y,\delta u}], \tag{1.5}$$

and $S_n(r)$ as

$$S_n(r) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\int \frac{\mathrm{d}\Omega}{4\pi}\int \frac{\mathrm{d}x}{L^3}|u(x+y,t)-u(x,t)|^n. \tag{1.6}$$

Note that these definitions of structure functions involve space-time-averaging, as well as an averaging over all possible orientations of y. Ensemble averages appear in Kolmogorov's original work. If the flow is homogeneous, space integration over x is unnecessary. If the flow is isotropic, the integration over solid angle Ω is redundant. If the flow is both homogeneous and isotropic, then the definition above will reduce to the usual ensemble average if it can be assumed that for stationary turbulence the system goes through all possible states over a long time. However, if the flow is neither homogeneous (say for an infinite geometry) nor known to be isotropic, the definitions of structure functions above are still meaningful and the results quoted in this paper remain valid. Earlier, Pullin & Saffman (1996), in the context of studying the Burger–Lundgren models for turbulence, also introduced spherical averaging, and noted that their spherically averaged structure functions satisfied the isotropic Karman–Howarth equation.

The incompressible forced Navier–Stokes equations determining the velocity field are given by

$$u_t(x,t) + u(x,t)\cdot\nabla u(x,t) = -\nabla p(x,t) + \nu\nabla^2 u(x,t) + f(x,t), \tag{1.7}$$

$$\nabla\cdot u(x,t) = 0. \tag{1.8}$$

It will be also assumed that the forcing $f(x,t)$ is such that the assumptions (a)–(c) listed below are valid:

(a) $\displaystyle\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\int\frac{\mathrm{d}x}{L^3}|f(x,t)|^2 \equiv \langle|f(x,t)|^2\rangle < \infty;$

(b) $\displaystyle\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\int\frac{\mathrm{d}x}{L^3}|\nabla f(x,t)|^2 \equiv \langle|\nabla f|^2\rangle < \infty;$

(c) $\displaystyle\bar{E} = \tfrac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\int\frac{\mathrm{d}x}{L^3}|u(x,t)|^2 \equiv \langle|u|^2\rangle < \infty.$

Note that we have introduced $\langle\cdot\rangle$ above to denote space-time-averaging, rather than the ensemble averages of Kolmogorov. It is to be noted that assumption (c) is only necessary for a completely unbounded geometry. For a periodic box (or even a strip) geometry, there exists a uniform upper bound for energy in terms of the forcing function f and the viscosity ν (Doering & Gibbon 1995). In that case, (c) actually follows from the given properties of f.

Because of assumptions (a)–(c) above, the characteristic energy-producing length-scale L can be precisely defined:

$$L = \frac{\langle|f|^2\rangle^{1/2}}{\langle|\nabla f|^2\rangle^{1/2}}. \tag{1.9}$$

We define a characteristic velocity scale U also in terms of the averaged energy $\bar{E}$, defined in (c):

$$U = \sqrt{2\bar{E}}. \tag{1.10}$$

The averaged dissipation, ϵ, is given by

$$\epsilon = \nu\langle|\nabla u|^2\rangle. \tag{1.11}$$

On taking the dot product of (1.7) with $u(x,t)$ and carrying out a space-time-integration, it follows that this average dissipation equals the averaged work done, i.e.

$$\epsilon = \langle f\cdot u\rangle. \tag{1.12}$$

It follows from Holder's inequality that

$$\epsilon \leqslant \langle |f|^2 \rangle^{1/2} U < \infty. \tag{1.13}$$

For the purposes of our paper, it is convenient to define two additional length-scales l and η_c as follows:

$$l = \frac{\epsilon}{U \langle |\nabla f|^2 \rangle^{1/2}}, \tag{1.14}$$

$$\eta_c = U(\nu/\epsilon)^{1/2}, \tag{1.15}$$

η_c is usually referred to in the literature as the Taylor microscale. It is to be noted from (1.9) and (1.13) that $l \leqslant L$.

Besides the assumptions (a)–(c), our results for $\tilde{S}_3(r)$ and $S_{3,K}$ have so far been shown to hold rigorously in the theoretical limit

$$\frac{\eta_c}{l} = \frac{\nu^{1/2} U^2 \langle |\nabla f|^2 \rangle^{1/2}}{\epsilon^{3/2}} \to 0. \tag{1.16}$$

For fixed forcing f in (1.7), this limit can be expected to be attained through a sequence of decreasing viscosity ν, or, perhaps, by increasing f for fixed ν. However, without knowledge of rigorous upper bounds on the energy $\bar{E}$ (and, therefore, U), and a suitable lower bound on the dissipation ϵ, we cannot rigorously show that the limit (1.16) is theoretically attainable by changing the given control parameters (viscosity ν and forcing f) in (1.7). Earlier, Kolmogorov hypothesized that the limiting ϵ as $\nu \to 0$ is finite and non-zero in three dimensions. If this can be assumed in addition to energy $\bar{E}$ growing slower than $\nu^{-1/2}$, then the theoretical limit (1.16) is implied by $\nu \to 0$. In the rigorous proofs presented here, we do not specifically assume the Kolmogorov hypotheses about ϵ or anything about the dependence of energy, $\bar{E}$, on viscosity. Our results only require the theoretical limit in equation (1.16). For the theoretical results to be physically valid, all we need to know is that the limit (1.16) is achievable, in principle, by suitably changing the control parameters of the problem. We will refer to the set of all distances r satisfying $\eta_c \ll r \ll l$, in the limit $\eta_c/l \to 0$ as an inertial scale, though the traditional definition of inertial scale is wider: $\eta \ll r \ll L$ (as the Reynolds number tends to infinity). We remark that our restrictions on inertial scale are needed for the purpose of achieving rigorous proofs; we fully expect the results to be valid over the wider traditional inertial scale. The reasons for this are discussed in the paper in remarks 2.4 and 2.5.

Under the assumptions (a)–(c) and the further assumption that solutions $u(x,t)$ to (1.7) and (1.8) exist for all times (a physically reasonable assumption that is yet to be proved rigorously), it will be proved that in the limit (1.16):

$$\text{(i)} \qquad \lim_{\substack{r/l \to 0,\\ r/\eta_c \to \infty}} (\tilde{S}_3(r)/\epsilon r) = -\tfrac{4}{3}; \tag{1.17}$$

$$\text{(ii)} \qquad \lim_{\substack{r/l \to 0,\\ r/\eta_c \to \infty}} (S_{3K}(r)/\epsilon r) = -\tfrac{4}{5}. \tag{1.18}$$

The latter relation (1.18) is the Kolmogorov four-fifths law. In the standard notation of asymptotics, (1.17) and (1.18) can, alternatively, be written as

$$\tilde{S}_3(r) \sim -\tfrac{4}{3}\epsilon r, \qquad S_{3K}(r) \sim -\tfrac{4}{5}\epsilon r, \tag{1.19}$$

for $l \gg r \gg \eta_c$. Since $S_3(r) \geqslant |\tilde{S}_3(r)|$, it follows from (1.17) that

$$\lim_{\substack{r/l \to 0, \\ r/\eta_c \to \infty}} (S_3(r)/\epsilon r) \geqslant \tfrac{4}{3}. \tag{1.20}$$

Further, as a consequence of (1.20), it follows from a routine application of the Holder inequality that for any $n > 3$ and $m < 3$,

$$\frac{S_n^{3-m}(r) S_m^{n-3}(r)}{[\epsilon r]^{n-m}} \geqslant (\tfrac{4}{3})^{n-m}, \tag{1.21}$$

with the inequality understood in the same sense as (1.20).

2. Derivation of results for $\tilde{S}_3(r)$

We now proceed to derive our results for $\tilde{S}_3$. We replace argument x by $x+y$ in (1.7) and subtract (1.7) from the new equation to obtain

$$[\delta u]_t + u(x,t) \cdot \nabla[\delta u] = -\nabla[\delta p] + \nu \nabla^2 [\delta u] + [\delta f] - (\partial / \partial y_j)[\delta u_j \delta u], \tag{2.1}$$

where

$$\delta u = u(x+y,t) - u(x,t), \qquad \delta p = p(x+y,t) - p(x,t), \qquad \delta f = f(x+y,t) - f(x,t), \tag{2.2}$$

and the subscript j denotes the jth component of the vector involved. A standard repeated index summation convention has also been used. A similar form of equations for vorticity appears in Constantin (1997). Taking the dot product of (2.1) with δu and integrating with respect to x over the entire volume (normalized by L^3), we obtain (after using (1.8) many times):

$$\frac{1}{2}\frac{\partial R}{\partial t}(y,t) + \nu \int \frac{\mathrm{d}x}{L^3} \nabla(\delta u_i) \cdot \nabla(\delta u_i) = \nabla \cdot N(y,t) + F(y,t). \tag{2.3}$$

In (2.3), the scalar functions $R(y,t)$ and $F(y,t)$ are defined as

$$R(y,t) = \int \frac{\mathrm{d}x}{L^3} |\delta u|^2, \tag{2.4}$$

$$F(y,t) = \int \frac{\mathrm{d}x}{L^3} (\delta f) \cdot (\delta u), \tag{2.5}$$

and the vector function $N(y,t)$ is given by

$$N(y,t) = -\frac{1}{2} \int \frac{\mathrm{d}x}{L^3} \delta u |\delta u|^2. \tag{2.6}$$

We notice that

$$\begin{aligned} \nu \int \frac{\mathrm{d}x}{L^3} \nabla(\delta u_i) \cdot \nabla(\delta u_i) &= 2\epsilon_1(t) - 2\nu \int \frac{\mathrm{d}x}{L^3} (\nabla u_i(x,t)) \cdot (\nabla u_i(x+y,t)) \\ &= 2\epsilon_1(t) - \nu \nabla^2 R, \end{aligned} \tag{2.7}$$

where $\epsilon_1(t)$ is the instantaneous normalized dissipation rate:

$$\epsilon_1(t) = \nu \int \frac{\mathrm{d}x}{L^3} |\nabla u(x,t)|^2. \tag{2.8}$$

Substituting (2.7) into (2.8), it is clear that

$$\frac{1}{2}\frac{\partial R}{\partial t}(y,t) + 2\epsilon_1(t) - \nu\nabla^2 R(y,t) = \nabla\cdot N(y,t) + F(y,t). \tag{2.9}$$

For isotropic flow, the dependence on y in (2.9) is only through $|y|$. In that case, (2.9) becomes the well-known Karman–Howarth equation. The anisotropic generalization in the form (2.9) appears to have been first derived by Monin & Yaglom (1975, p. 402). Here, we have re-derived this for the sake of completeness in the present context (where space-time-averaging replaces ensemble averaging of an assumed statistically stationary homogeneous flow). Also, some of the intermediate steps leading up to (2.9) are useful in our later derivation of the Kolmogorov four-fifths law, without the Kolmogorov assumption on isotropy.

We replace y by $\tilde{y}$ in (2.9) and integrate with respect to $\tilde{y}$ over a sphere of radius $r = |y|$ centred at $\tilde{y} = 0$. Using divergence theorem on resulting volume integrals, and dividing the result by $2\pi r^2$, we obtain

$$\frac{\partial}{\partial t}\left\{\frac{1}{4\pi r^2}\int_{|\tilde{y}|<r} \mathrm{d}\tilde{y} R(\tilde{y},t)\right\} + \tfrac{4}{3}\epsilon_1 r - 2\nu\frac{\mathrm{d}}{\mathrm{d}r}T_2(r,t) - \frac{1}{2\pi r^2}\int_{|\tilde{y}|<r}\mathrm{d}\tilde{y}F(\tilde{y},t) = -\tilde{T}_3(r,t), \tag{2.10}$$

where $T_2(r,t)$ is defined by expression (1.6) for $n = 2$, but without any time-averaging. Similarly, $\tilde{T}_3(r,t)$ is defined by the same expression as for $\tilde{S}_3$ in (1.4), except that time-averaging is not performed. On time-integrating (2.10) from 0 to T and dividing the resulting expression by T, and taking the limit as $T\to\infty$, it follows that

$$\tfrac{4}{3}\epsilon r - 2\nu\frac{\mathrm{d}}{\mathrm{d}r}S_2(r) - \frac{1}{2\pi r^2}\int_{|\tilde{y}|<r}\mathrm{d}\tilde{y}\,G(\tilde{y}) = -\tilde{S}_3(r), \tag{2.11}$$

where

$$\epsilon = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\epsilon_1(t), \qquad G(\tilde{y}) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}tF(\tilde{y},t). \tag{2.12}$$

So far, everything is exact and involves no approximations or assumptions on the nature of the flow or the range of r.

We now present two propositions that ensure that the second and third terms on the left-hand side of (2.11) are asymptotically negligible compared to ϵr in the asymptotic limit (1.16), when $l \gg r \gg \eta_c$.

Proposition 2.1. *Under assumptions (a)–(c), if a smooth solution*† $u(x,t)$ *satisfying (1.7) and (1.8) exists for all times, then*

$$\lim_{r/l\to 0}\frac{1}{\epsilon r^3}\int_{|\tilde{y}|<r}\mathrm{d}\tilde{y}\,G(\tilde{y},t) = 0. \tag{2.13}$$

Proof. We note that on using the well-known triangular equality

$$|\delta u\cdot\delta f| \leqslant [|u(x+y,t)| + |u(x,t)|]|\delta f|.$$

† It was pointed out by P. Constantin (personal communication) that the condensing of propositions 2.1, 2.2, 3.1 and 3.2 is valid, even for a known global weak solution.

On using $\|\delta f\| \leqslant \|\nabla f\| |y|$, where $\|\cdot\|$ denotes the $\mathcal{L}_2$ norm in x, and Holder's inequality, it follows that

$$|G(\tilde{y})| \leqslant 2U\langle|\nabla f(x,t)|^2\rangle^{1/2}|\tilde{y}|. \tag{2.14}$$

It follows that

$$\left|\frac{1}{2\pi r^2}\int_{|\tilde{y}|<r} \mathrm{d}\tilde{y}\, G(\tilde{y})\right| \leqslant Ur^2\langle|\nabla f(x,t)|^2\rangle^{1/2}. \tag{2.15}$$

On dividing (2.15) by ϵr and using the definition of l from (1.14), the statement of proposition 2.1 follows. ■

Proposition 2.2. *Under assumptions (a)–(c), if a smooth solution $u(x,t)$ satisfying (1.5) and (1.6) exists for all times, then*

$$\lim_{r/\eta_{\mathrm{c}}\to\infty} \frac{\nu}{\epsilon r}\frac{\mathrm{d}}{\mathrm{d}r}S_2(r) = 0. \tag{2.16}$$

Proof. First, we note that

$$\nabla_y \int \mathrm{d}x\,|\delta u|^2 = \int \mathrm{d}x\,(\delta u_i)\nabla u_i(x+y,t) = \int \mathrm{d}x\,(u_i(x,t) - u_i(x-y,t))\nabla u_i(x,t), \tag{2.17}$$

where the symbol ∇_y is the gradient with respect to the y variable. Therefore, using (2.17),

$$\left|\frac{y}{r}\cdot\nabla_y \int \frac{\mathrm{d}x}{L^3}(|\delta u|^2)\right| \leqslant \int \frac{\mathrm{d}x}{L^3}|u(x)|\,|\nabla u(x,t)| + \int \frac{\mathrm{d}x}{L^3}|u(x-y)|\,|\nabla u(x,t)|. \tag{2.18}$$

We notice from (1.6) that $S_2'(r)$ is bounded in absolute value by the time-solid-angle average of the expression on the left-hand side of (2.18), which, from Holder's inequality, is bounded by

$$\leqslant 2\langle|u(x,t)|^2\rangle^{1/2}\langle|\nabla u(x,t)|^2\rangle^{1/2}.$$

Therefore,

$$\nu|S_2'(r)| \leqslant 2U\nu\langle|\nabla u(x,t)|^2\rangle^{1/2} = 2U\nu^{1/2}\epsilon^{1/2}. \tag{2.19}$$

Proposition 2.2 follows from (2.19), if we divide it by ϵr and use the definition of η_{c} in (1.15). ■

Remark 2.3. Both limits on r appearing in (2.13) and (2.16) can be satisfied if and only if $\eta_{\mathrm{c}}/l \to 0$, as assumed in (1.16).

Remark 2.4. $r/\eta_{\mathrm{c}} \to 0$ is a sufficient condition for the conclusion of proposition 2.2 to hold. It is not expected to be necessary. Indeed, if the Kolmogorov expression (1.2) is assumed valid for $S_2(r)$ and used to evaluate $S_2'(r)$ in (2.16), then it is clear that (2.16) would remain equally valid for $r/\eta \to 0$ for $\eta = \nu^{3/4}\epsilon^{-1/4}$.

Remark 2.5. There is no particular physical reason to expect that the magnitude of the dot product average $\langle f\cdot u\rangle$ should not be of the same order as $\langle|f|^2\rangle^{1/2}U$. In that case, $\epsilon/(U\langle|f|^2\rangle^{1/2})$ is strictly $O(1)$. l is then strictly the same order as L. Combining this with remark 2.4, we can expect that (2.13) and (2.16) are actually valid over the entire inertial regime: $\eta \ll r \ll L$, though the rigorous proofs so far are limited to the subrange $\eta_{\mathrm{c}} \ll r \ll l$.

3. A derivation of the Kolmogorov four-fifths law

We now proceed to use the result (1.17) for $\tilde{S}_3(r)$ to re-derive the Kolmogorov four-fifths law, but without the Kolmogorov assumption on flow isotropy. For this purpose, it is convenient to return to (2.1) and take the dot product with y. This leads to

$$[y \cdot \delta u]_t + u(x,t) \cdot \nabla [y \cdot \delta u] = -\frac{\partial}{\partial x_i}[y_i \delta p] + \nu \nabla^2 [y \cdot \delta u] + [y \cdot \delta f] - \frac{\partial}{\partial y_j}[\delta u_j \delta u_i y_i] + [\delta u_j \delta u_j]. \quad (3.1)$$

On multiplying (3.1) by $y \cdot \delta u$, integrating with respect to x over the whole volume, replacing y by $\tilde{y}$ and integrating with respect to $\tilde{y}$ over a sphere of radius $r = |y|$ in a manner similar to that shown explicitly in (2.1)–(2.10), we get

$$\frac{\partial}{\partial t}\left[2\pi \int_0^r \mathrm{d}\tilde{r}\tilde{r}^4 T_{2K}(\tilde{r},t)\right] + \nu Q = P - 2\pi r^4 T_{3K}(r,t) + 4\pi \int_0^r \tilde{T}_3(\tilde{r},t)\tilde{r}^3 \,\mathrm{d}\tilde{r} + \int_{|\tilde{y}|<r} \mathrm{d}\tilde{y}\, \hat{F}(\tilde{y},t), \quad (3.2)$$

where T_{2K} and T_{3K} are given by the expression (1.5) (for $n = 2$ and $n = 3$), except that integration with respect to t is not performed. In (3.2), Q, P and $\hat{F}$ are defined by

$$Q(r,t) = \int_{|\tilde{y}|<r} \mathrm{d}\tilde{y} \int \frac{\mathrm{d}x}{L^3} [\tilde{\delta u}]_{k,j} [\tilde{\delta u}]_{l,j} \tilde{y}_k \tilde{y}_l, \quad (3.3)$$

$$P(r,t) = \int_{|\tilde{y}|<r} \mathrm{d}\tilde{y} \int \frac{\mathrm{d}x}{L^3} ([\tilde{\delta p}][\tilde{\delta u}]_{k,i} \tilde{y}_i \tilde{y}_k), \quad (3.4)$$

$$\hat{F}(\tilde{y},t) = \int \frac{\mathrm{d}x}{L^3} [\tilde{\delta u}_k] \tilde{y}_k [\tilde{\delta f}_i] \tilde{y}_i, \quad (3.5)$$

where the subscript '$,j$' refers to differentiation with respect to x_j, and $\tilde{\delta u}$ refers to δu with argument y replaced by $\tilde{y}$ in (2.2). (Similarly, for $\tilde{\delta p}$, etc.)

After some manipulation, and using

$$\int_{|\tilde{y}|<r} \mathrm{d}\tilde{y}\, \tilde{y}_k \tilde{y}_l = \tfrac{4}{15}\pi r^5 \delta_{k,l}, \quad (3.6)$$

where $\delta_{k,l}$ stands for the usual Kronecker delta, we find

$$\nu Q(r,t) = \tfrac{8}{15}\pi \epsilon_1(t) r^5 - 2\nu Q_1(r,t), \quad (3.7)$$

where

$$Q_1(r,t) = \int_{|\tilde{y}|<r} \mathrm{d}\tilde{y} \int \frac{\mathrm{d}x}{L^3} u_{l,j}(x,t) u_{k,j}(x+\tilde{y},t) \tilde{y}_l \tilde{y}_k. \quad (3.8)$$

On integrating by parts with respect to x, and then with respect to $\tilde{y}$, one obtains

$$Q_1(r,t) = -\int \frac{\mathrm{d}x}{L^3} u_l(x,t) \int \mathrm{d}\Omega\, y_j y_l y_k r u_{k,j}(x+y,t) + \int \frac{\mathrm{d}x}{L^3} u_j(x,t) \int_{|\tilde{y}|<r} \mathrm{d}\tilde{y}\, \tilde{y}_k u_{k,j}(x+\tilde{y},t). \quad (3.9)$$

The latter integral term in (3.9) is zero from integration by parts with respect to x. Also notice, that in the first integral,

$$y_j u_{k,j}(x+y,t) = r(\partial/\partial r)u_k(x+y,t).$$

Therefore, it follows from (3.7) and (3.9) that

$$\nu Q(r,t) = \tfrac{8}{15}\pi\epsilon_1(t)r^5 - 4\pi\nu r^4 T'_{2K}(r,t), \tag{3.10}$$

where the prime denotes differentiation with respect to r.

We will now show that $P=0$. For this purpose, we notice that P can be written as

$$P(r,t) = 2\int\frac{\mathrm{d}x}{L^3}\int_{|\tilde{y}|<r}\mathrm{d}\tilde{y}\,p(x,t)u_{k,i}(x,t)\tilde{y}_i\tilde{y}_k - 2\int\frac{\mathrm{d}x}{L^3}\int_{|\tilde{y}|<r}\mathrm{d}\tilde{y}\,p(x,t)u_{k,i}(x+\tilde{y},t)\tilde{y}_i\tilde{y}_k. \tag{3.11}$$

On using (3.6), and the divergence condition $u_{k,k}=0$, it follows that the first integral on the right-hand side of (3.11) is zero. On integrating the second integral by parts with respect to $\tilde{y}$, we get

$$P(r,t) = -2r^3\int\frac{\mathrm{d}x}{L^3}\int\mathrm{d}\Omega\,p(x,t)u_k(x+y,t)y_k + 8\int\frac{\mathrm{d}x}{L^3}\int_{|\tilde{y}|<r}\mathrm{d}\tilde{y}\,p(x,t)u_k(x+\tilde{y},t)\tilde{y}_k. \tag{3.12}$$

The first integral in (3.12) is clearly zero since the surface integral

$$\int\mathrm{d}\Omega\,u_k(x+y,t)y_k = \frac{1}{r}\int_{|\tilde{y}|<r}\mathrm{d}\tilde{y}\,u_{k,k}(x+\tilde{y},t) = 0. \tag{3.13}$$

We also notice that

$$2\tilde{y}_k = \frac{\partial}{\partial\tilde{y}_k}\tilde{r}^2,$$

where $\tilde{r}^2 = \tilde{y}_j\tilde{y}_j$. Using this, the second integral in (3.12) can be integrated by parts with respect to $\tilde{y}$ and using (3.13) again, we get this to be zero as well. Thus $P=0$.

Using the simplifications for Q and P back in (3.2) and time-integrating this equation from 0 to T, and then dividing it by T, we obtain, in the limit $T\to\infty$ (after dividing by $2\pi r^4$),

$$\tfrac{4}{15}\epsilon r - \frac{2}{r^4}\int_0^r\tilde{S}_3(\tilde{r})\tilde{r}^3\,\mathrm{d}\tilde{r} - 2\nu S'_{2K}(r) = -S_{3K}(r) + \frac{1}{2\pi r^4}\int_{|\tilde{y}|<r}\mathrm{d}\tilde{y}\,\hat{G}(\tilde{y}), \tag{3.14}$$

where

$$\hat{G}(\tilde{y}) = \lim_{T\to\infty}\frac{1}{T}\int_0^T\mathrm{d}t\hat{F}(\tilde{y},t). \tag{3.15}$$

We now claim that in the inertial regime, as identified before, $\nu T'_2(r)$ and

$$\frac{1}{r^4}\int_{|\tilde{y}|<r}\mathrm{d}\tilde{y}\,\hat{G}(\tilde{y})$$

are negligible, compared to ϵr in (3.14). We make the following propositions, whose proofs closely parallel those of propositions 2.1 and 2.2.

Proposition 3.1. *If a smooth solution $u(x,t)$ satisfying (1.5) and (1.6) exists for all times, then*

$$\lim_{r/l\to 0}\frac{1}{\epsilon r^5}\int_{|\tilde{y}|<r}\mathrm{d}\tilde{y}\,\hat{G}(\tilde{y})=0. \quad (3.16)$$

Proof. We note, on using the well-known triangular equality, that

$$|[\tilde{y}\cdot\tilde{\delta}u][\tilde{y}\cdot\tilde{\delta}f]|\leqslant 2|\tilde{y}|^2[|u(x+\tilde{y},t)|+|u(x,t)|]|\tilde{\delta}f|.$$

On using $\|\tilde{\delta}f\|\leqslant\|\nabla f\|\,|\tilde{y}|$, where $\|\cdot\|$ denotes the $\mathcal{L}_2$ norm in x, and Holder's inequality, it follows that

$$|\hat{G}(\tilde{y})|\leqslant 2\langle u^2(x,t)\rangle^{1/2}\langle|\nabla f(x,t)|^2\rangle^{1/2}|\tilde{y}|^3. \quad (3.17)$$

It follows that

$$\left|\frac{1}{\pi r^4}\int_{|\tilde{y}|<r}\mathrm{d}\tilde{y}\,\hat{G}(\tilde{y})\right|\leqslant\tfrac{4}{3}\langle|\nabla f(x,t)|^2\rangle^{1/2}Ur^2. \quad (3.18)$$

On dividing (2.15) by ϵr and using the definition of l from (1.14), the statement of proposition 3.1 follows. ■

Proposition 3.2. *If a smooth solution $u(x,t)$ satisfying (1.5) and (1.6) exists for all times, then*

$$\lim_{r/\eta_{\mathrm{c}}\to\infty}\frac{\nu}{\epsilon r}\frac{\mathrm{d}}{\mathrm{d}r}S_{2K}(r)=0. \quad (3.19)$$

Proof. First, we note that $S'_{2K}(r)$ is the time-average of $T'_{2K}(r,t)$. From (3.7) and (3.10), it clearly follows that

$$T'_{2K}(r,t)=\frac{1}{2\pi r^4}Q_1(r,t). \quad (3.20)$$

But, from expression (3.9) for $Q_1(r,t)$, we know that only the first integral is non-zero and this, after integration by parts with respect to x, leads to

$$T'_{2K}(r,t)=\frac{1}{2\pi r^4}\int\frac{\mathrm{d}x}{L^3}u_{l,j}(x,t)\int\mathrm{d}\Omega\,y_l y_j y_k r u_k(x+y,t). \quad (3.21)$$

Noticing that each of y_l, y_j and y_k are bounded by r, on long time-integration of the above equation and using Holder's inequality, it follows that

$$\nu|S'_{2K}(r)|\leqslant 2\nu U\langle|\nabla u|^2\rangle^{1/2}=2U\nu^{1/2}\epsilon^{1/2}. \quad (3.22)$$

Proposition 3.2 follows by dividing the above expression by ϵr, and using the definition of η_{c} in (1.15). ■

Remark 3.3. As before, with $\tilde{S}_3(r)$, propositions 3.1 and 3.2 both hold when $\eta_{\mathrm{c}}\ll r\ll l$. This defines the inertial scale for the purposes of the proof, though a wider range, $\eta\ll r\ll L$, is expected, as discussed in remarks 2.4 and 2.5.

Given propositions 3.1 and 3.2, it implies that in the inertial range, (3.14) simplifies to

$$\tfrac{4}{15}\epsilon r-\frac{2}{r^4}\int_0^r\tilde{S}_3(\tilde{r})\tilde{r}^3\,\mathrm{d}\tilde{r}=-S_{3K}(r), \quad (3.23)$$

where the equality in (3.23) holds in the same sense as (1.17). Using the relation (1.17), proved before for $r/\eta_c \gg 1$ and $r/l \ll 1$, it is not difficult to see that in the inertial range,

$$-\frac{2}{r^4}\int_0^r \tilde{S}_3(\tilde{r})\tilde{r}^3\,\mathrm{d}\tilde{r} = \tfrac{8}{15}\epsilon r. \tag{3.24}$$

The simplest way to prove this result is to institute a change in the variable $s = r/\eta$, $\tilde{s} = \tilde{r}/\eta$, so that the left-hand side of the above equation becomes

$$-\frac{2}{s^4}\int_0^s \mathrm{d}\tilde{s}\tilde{S}_3(\tilde{s}\eta)\tilde{s}^3\,\mathrm{d}\tilde{s}. \tag{3.25}$$

Since $s \gg 1$, it is clear from (1.17), that the leading asymptotic behaviour of the integral above for large s is dominated by the contribution near the upper limit. This gives the result (3.24), where the equality is to be understood in the asymptotic sense as in (1.17). Using the result (3.24) in (3.23), we obtain the Kolmogorov four-fifths law, given by (1.18). Hence the proof is complete.

4. Numerical simulation

Since the computation of $\tilde{S}_3$ and $S_{3,K}$ involves one three-dimensional space integration with respect to x, a two-dimensional solid angle averaging with respect to orientations of y, as well as a one-dimensional time-integration, numerical computations for $\tilde{S}_3$ and $S_{3,K}$ for sufficiently large Reynolds number were deemed prohibitively expensive.

Instead, we computed $\tilde{S}_3^c$ and $S_{3,K}^c$, defined as

$$\tilde{S}_3^c = \frac{1}{T}\int_0^T \mathrm{d}t \int \frac{\mathrm{d}x}{L^3}|u(x+y,t)-u(x,t)|^3\cos\theta_{\delta u,y}, \tag{4.1}$$

$$S_{3,K}^c = \frac{1}{T}\int_0^T \mathrm{d}t \int \frac{\mathrm{d}x}{L^3}|u(x+y,t)-u(x,t)|^3\cos^3\theta_{\delta u,y}, \tag{4.2}$$

which are, respectively, modifications of $\tilde{S}_3$ and $S_{3,K}$, in that no solid angle averaging is involved. We chose two significantly different orientations of the vector y: $y = (r/\sqrt{3})(1,1,1)$ and $y = r(1,0,0)$. At the outset, we expected that for a highly symmetric flow, the assumptions of isotropy would actually be satisfied over some range of scales at the highest computable Reynolds number. This would make solid angle averaging moot. This would be suggested by the independence of computed $\tilde{S}_3^c/(\epsilon r)$ and $S_{3,K}^c/(\epsilon r)$ on the orientations of y. However, this did not turn out to be the case. The numerical results are presented here to indicate the degree of anisotropy of the flow and how well the linear scaling of the third-order structure functions hold in some regime of r for specific orientations of y.

We solve (1.7)–(1.8) in a 2π-periodic cube with an initial condition of '*high symmetry*' as discussed in Kida (1985). In particular, the flow at all times admits the following Fourier expansion for the x_1 component of the velocity at all times:

$$u_1(x_1,x_2,x_3,t) = \left(\sum_{\text{even } l,m,n=0}^{\infty} + \sum_{\text{odd } l,m,n=1}^{\infty}\right)\hat{u}_{1\{l,m,n\}}(t)\sin lx_1\cos mx_2\cos nx_3. \tag{4.3}$$

The other velocity components are determined by the permutation symmetry

$$u_1(x_1, x_2, x_3) = u_2(x_3, x_1, x_2) = u_3(x_2, x_3, x_1).$$

The special structure of the Fourier components in (4.1) and the permutation relationship above saves computational time and memory (Kida 1985; Boratav & Pelz 1994). In our study, the initial condition is chosen to be the same as that in Kida *et al.* (1989) and Kida & Ohkitani (1992). Specifically,

$$u_1(x_1, x_2, x_3, t = 0) = \sin x_1(\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3). \tag{4.4}$$

Two kinds of forcing, $f(x, t)$, are used in this study:

(a) $f_1(x_1, x_2, x_3, t) = u_1(x_1, x_2, x_3, t = 0)/5$;

(b) $f(x, t)$ is chosen such that the Fourier mode $\hat{u}_{1\{1,3,1\}} = -\hat{u}_{1\{1,1,3\}} = 1$ all the time, in order to imitate a constant energy supply at lower wavenumbers (Kida *et al.* 1989; Kida & Ohkitani 1992). No forcing is used in any of the other wavenumbers. From (1.7), this is equivalent to choosing the x_1 component of the force as

$$f_1(x, t) = \mathcal{P}[u_{1t} + u \cdot \nabla u_1 + (\partial p/\partial x_1) + \nu\nabla^2 u_1], \tag{4.5}$$

where $\mathcal{P}$ is the projection to the space of scalar functions generated by the basis

$$\{\sin x_1 \cos[3x_2] \cos x_3,\ \sin x_1 \cos x_2 \cos[3x_3]\}, \tag{4.6}$$

where $\mathcal{P}u_1$ is given as

$$\mathcal{P}u_1(x_1, x_2, x_3, t) = \sin x_1 \cos[3x_2] \cos x_3 - \sin x_1 \cos x_2 \cos[3x_3]. \tag{4.7}$$

The other components f_2, f_3 of f are similarly determined from the cyclic permutation property of u_2 and u_3.

The numerical method for solving (1.7)–(1.8) is based on a Fourier pseudo-spectral technique. The details can be found in Boratev *et al.* (1992) and Boratav & Pelz (1994). To perform the integration in time for $\tilde{S}_3^c$, $S_{3,K}^c$ and ϵ, we use a second-order Adams–Bashforth method. For the sake of saving computational time, the time-step for the integration is chosen to be $5\Delta t$, where Δt is the time-step for solving the corresponding Navier–Stokes equations (1.7)–(1.8). The spatial integrations in x for $\tilde{S}_3^c$, $S_{3,K}^c$ and ϵ are evaluated through summation over $N^3/64$ evenly spaced grid points in the 2π-periodic box, where N is the number of grid points in each direction of the 2π-periodic domain. This quadrature is spectrally accurate. For a computational reason, we always choose y in $\tilde{S}_3^c$, $S_{3,K}^c$ on the grid points, or the periodic extension of the grid points.

For $f(x, t) = 0$, we have tested our computational results against those presented in Boratav & Pelz (1994). For the forcing (b), we have compared our results with those studied in Kida *et al.* (1989) for large ν, for example, $\nu = 0.011$. We have also performed resolution studies in N and Δt for the computations of $\tilde{S}_3^c$, $S_{3,K}^c$ and ϵ. All computations are performed by using 64 bit arithmetic.

In this study, we are interested in the following quantities:

$$G_1 \equiv -\frac{\tilde{S}_3^c}{r\epsilon}, \qquad G_2 \equiv -\frac{S_{3,K}^c}{r\epsilon}, \tag{4.8}$$

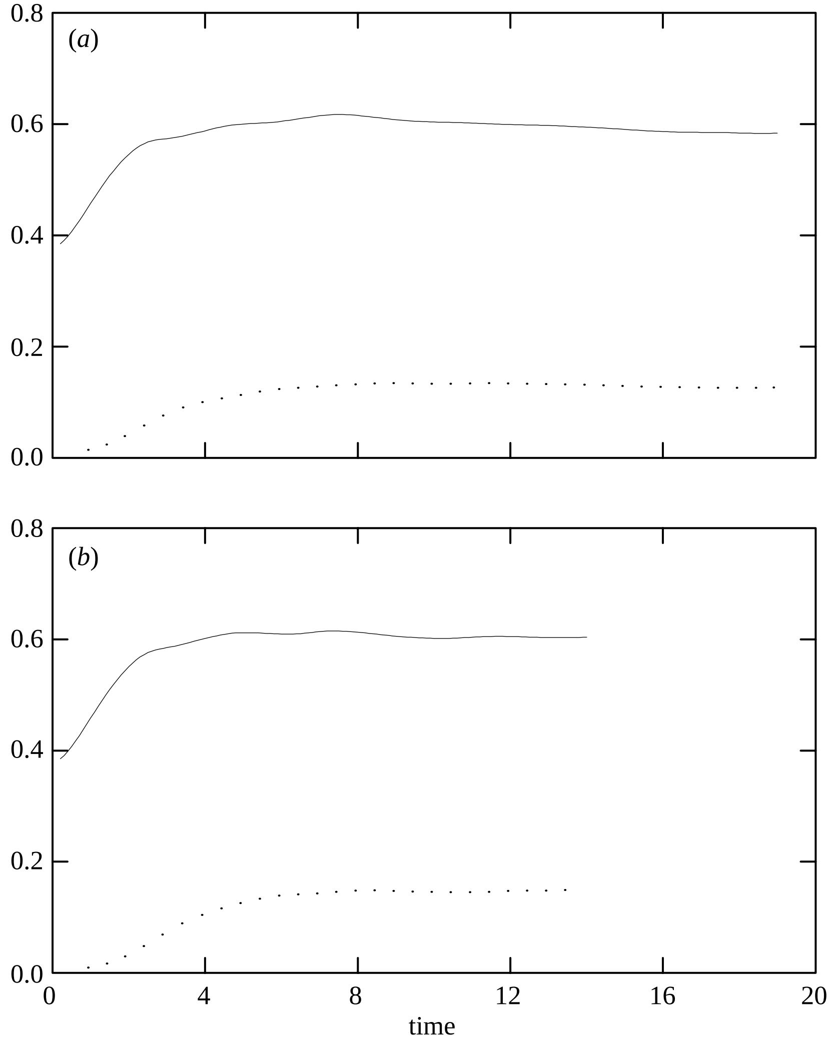

Figure 1. Averaged normalized energy versus time for forcing (a) (——). Averaged normalized energy dissipation rate versus time for forcing (a) (· · · · ·). (*a*) $\nu = 0.001$; (*b*) $\nu = 0.000\,667$.

as functions of r/η for various r. It was rigorously proved in the previous section that as $r/l \to 0$ and $r/\eta_c \to \infty$, $G_1 \to \frac{4}{3}$ and $G_2 \to \frac{4}{5}$ for isotropic flows. Nonetheless, in view of earlier remarks, we expect the results to be valid in the full inertial range: $\eta \ll r \ll L$ in the limit $\eta/L \to 0$. However, in our numerical computations for finite non-zero ν, η/L is a non-zero-though-small number, and G_1 and G_2 are generally functions of r, orientation of y, as well as T—the time-integration length.

We study the forcing (a) from figure 1 to figure 3. In figure 1, we plot averaged normalized averaged energy $\bar{E} = \langle |u(x,t)|^2 \rangle$ and averaged normalized energy dissipation rate ϵ as functions of T for $\nu = 0.001$ and $\nu = 0.000\,667$. L, the reference length-scale is chosen to be 2π for computational reasons, rather than that given by (1.9). This differing choice, made for the sake of simplicity, makes no difference to

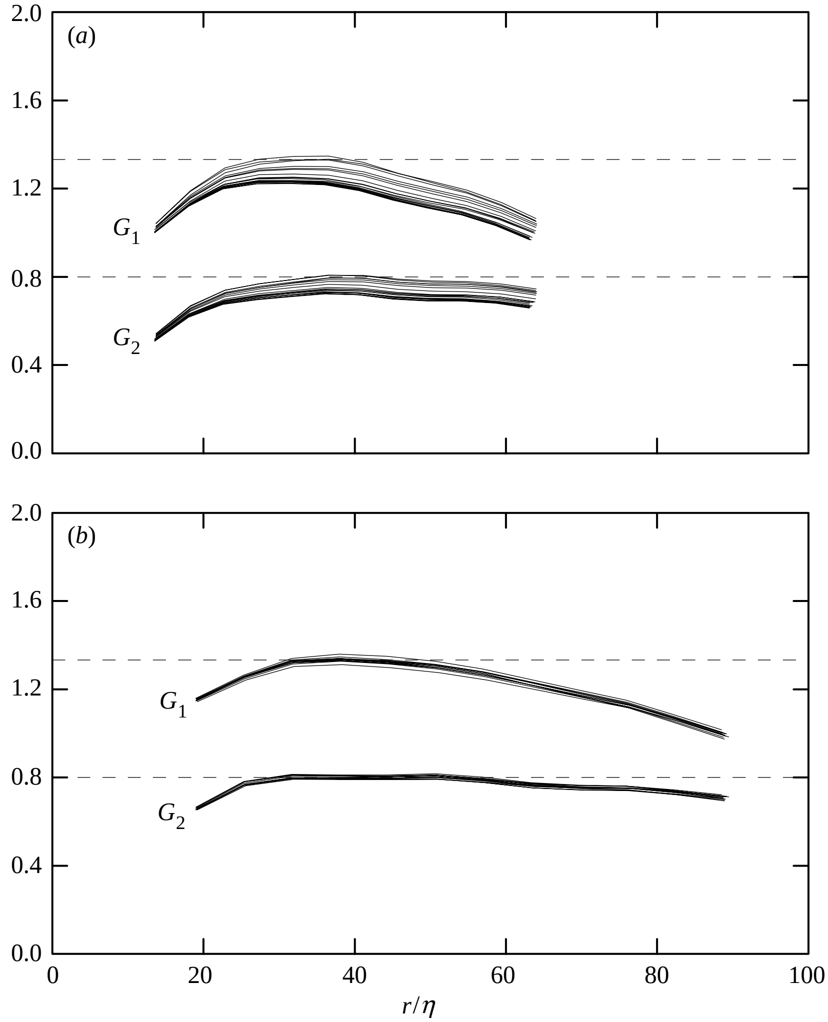

Figure 2. G_1 and G_2 as functions of r/η at different T for $y = (r/\sqrt{3})(1,1,1)$ for forcing (a). $\frac{4}{3}$ or $\frac{4}{5}$ (– – –). (*a*) $\nu = 0.001$ and $T = 10 + i \times 0.5$ for $i = 0, \dots, 18$; (*b*) $\nu = 0.000\,667$ and $T = 9 + i \times 0.5$ for $i = 0, \dots, 10$. Note convergence for large enough T.

computations of G_1 and G_2, since their dependence on L cancels out. Also, $L = 2\pi$ is clearly only a multiple of L given by (1.9), with the multiple independent of ν.

Here, we choose $N = 256$ for both cases with $\Delta t = 0.001$ for $\nu = 0.001$ and $\Delta t = 0.0005$ for $\nu = 0.000\,667$ (Kida *et al.* 1989; Boratav & Pelz 1994). It appears that $\bar{E}$ and ϵ start to settle down around $T = 10$ for $\nu = 0.001$ and $T = 9$ for $\nu = 0.000\,667$, respectively. Here the time-scale is implicit by the choice $L = 2\pi$, and taking f, which has units of acceleration, to be given by (a). Based on the equilibrated values of $\bar{E}$ and ϵ, a Taylor microscale Reynolds number Re_λ is defined as

$$Re_\lambda = \sqrt{\frac{20}{3}}\frac{\bar{E}}{\sqrt{\nu\epsilon}}. \tag{4.9}$$

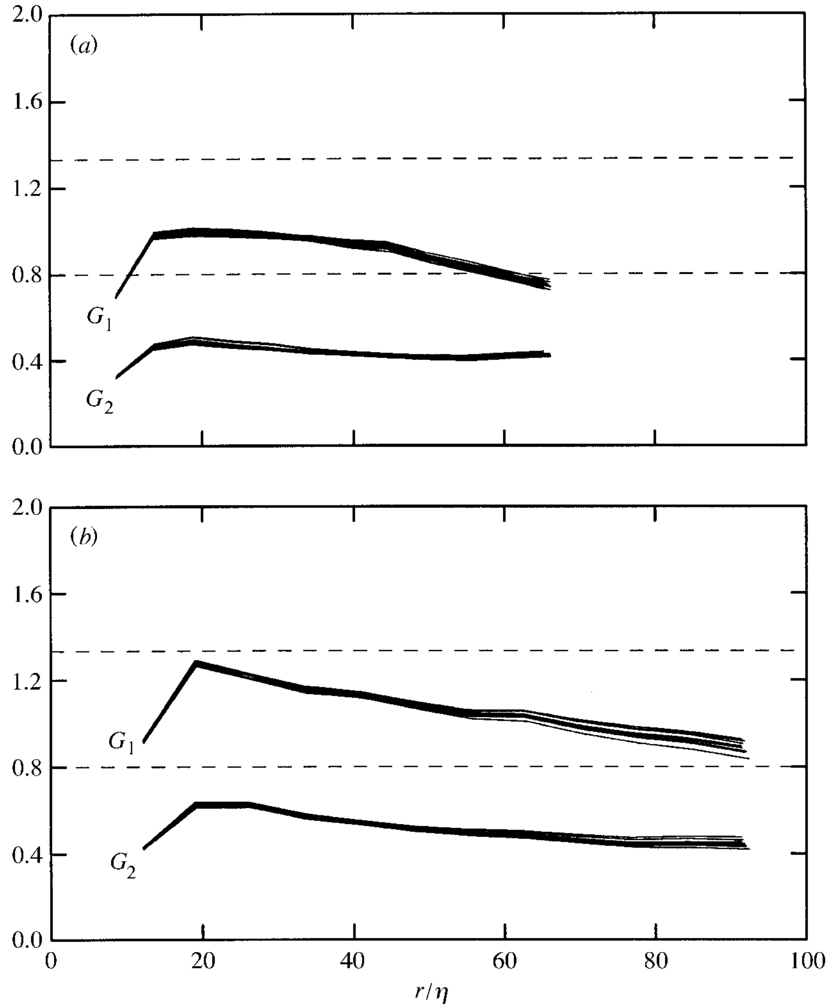

Figure 3. G_1 and G_2 as functions of r/η at different T for $y = (r, 0, 0)$ for forcing (a). $\frac{4}{3}$ or $\frac{4}{5}$ (– – –). (*a*) $\nu = 0.001$ and $T = 10 + i \times 0.5$ for $i = 0, \dots, 18$; (*b*) $\nu = 0.000\,667$ and $T = 9 + i \times 0.5$ for $i = 0, \dots, 10$. Note convergence for large enough T.

It is known that for sufficiently large values, this Taylor Reynolds number, Re_λ, scales as the square root of the Reynolds number Re. The $\nu = 0.001$ calculation reported here corresponds to $Re_\lambda = 132$, while for the $\nu = 0.000\,667$ calculation, $Re_\lambda = 154$. The corresponding values of η/L turned out to be 0.001 49 and 0.001 06, respectively, while η_c/η equalled 10.09 and 10.95, respectively.

In figure 2, we plot G_1 and G_2 against r/η for different T for the same cases shown in figure 1. Here, we choose $y = (r/\sqrt{3})(1, 1, 1)$, where $r = |y|$ is given. As shown in the graphs, there is a range of r (though not very large) where G_2 is approximately $\frac{4}{5}$, while G_1 is approximately $\frac{4}{3}$. The agreement for G_2 appears to be better. Given the theoretical results, it is not surprising that the range of r over which G_1 and G_2 are approximately $\frac{4}{3}$ and $\frac{4}{5}$ is larger for $Re_\lambda = 154$ than that for $Re_\lambda = 132$. Indeed, the fit with a constant is also better for the larger Re_λ.

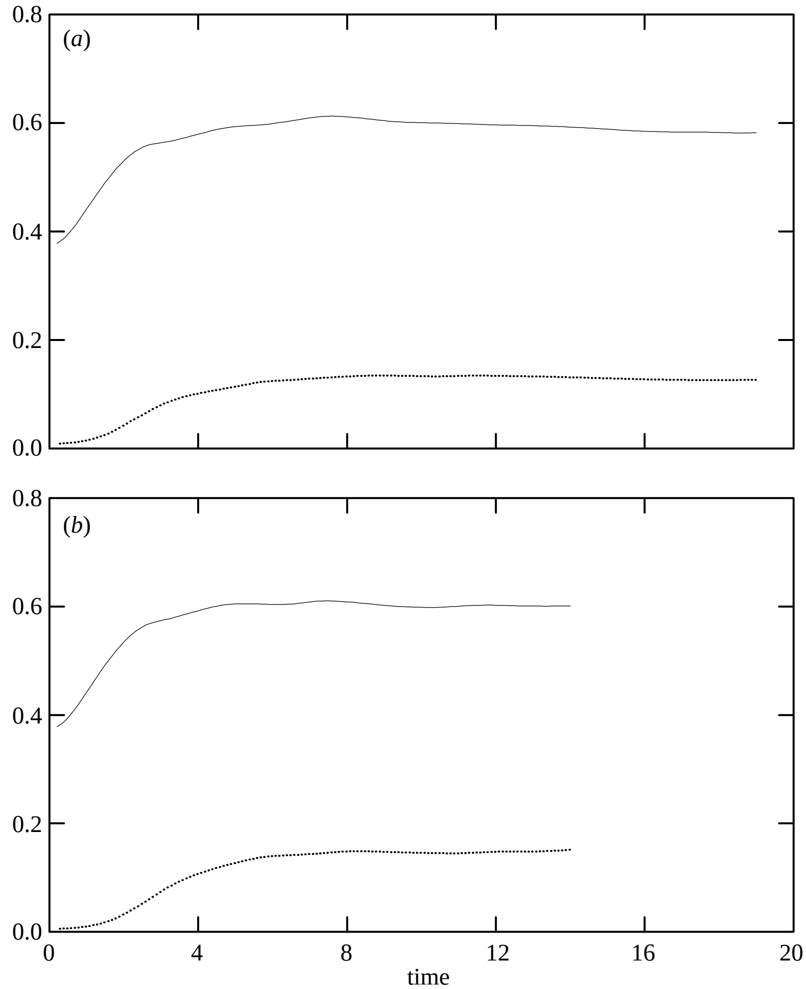

Figure 4. Averaged normalized energy versus time for forcing (b) (——). Averaged normalized energy dissipation rate versus time for forcing (b) (· · · · ·). (*a*) $\nu = 0.001$; (*b*) $\nu = 0.000\,667$.

Similar results are presented in figure 3 for $y = r(1, 0, 0)$ with r given. We see that there is no significant regime of r/η where G_1 and G_2 are constants. Further, the values are significantly smaller than $\frac{4}{3}$ and $\frac{4}{5}$, respectively, though the values are somewhat larger for $\nu = 0.000\,67$ than for $\nu = 0.001$. *It is possible that these approach* $\frac{4}{3}$ *and* $\frac{4}{5}$*, respectively, as* ν *becomes even smaller.*

A similar study has been performed for forcing (b), where the $\nu = 0.001$ calculation corresponds to $Re_\lambda = 134$, while for the $\nu = 0.000\,667$ calculation, $Re_\lambda = 155$. Similar quantities to those in figures 1–3 are plotted in figures 4–6. It is easy to see that the corresponding graphs are very similar to each other.

The significant differences between figures 2 and 3 as well as between figures 5 and 6 suggest a lack of isotropy in the flow. This is consistent with earlier numerical calculations (Yeung *et al.* 1995) that suggest that third-order moments do remain anisotropic even at later times. While the theoretically predicted quantities involve

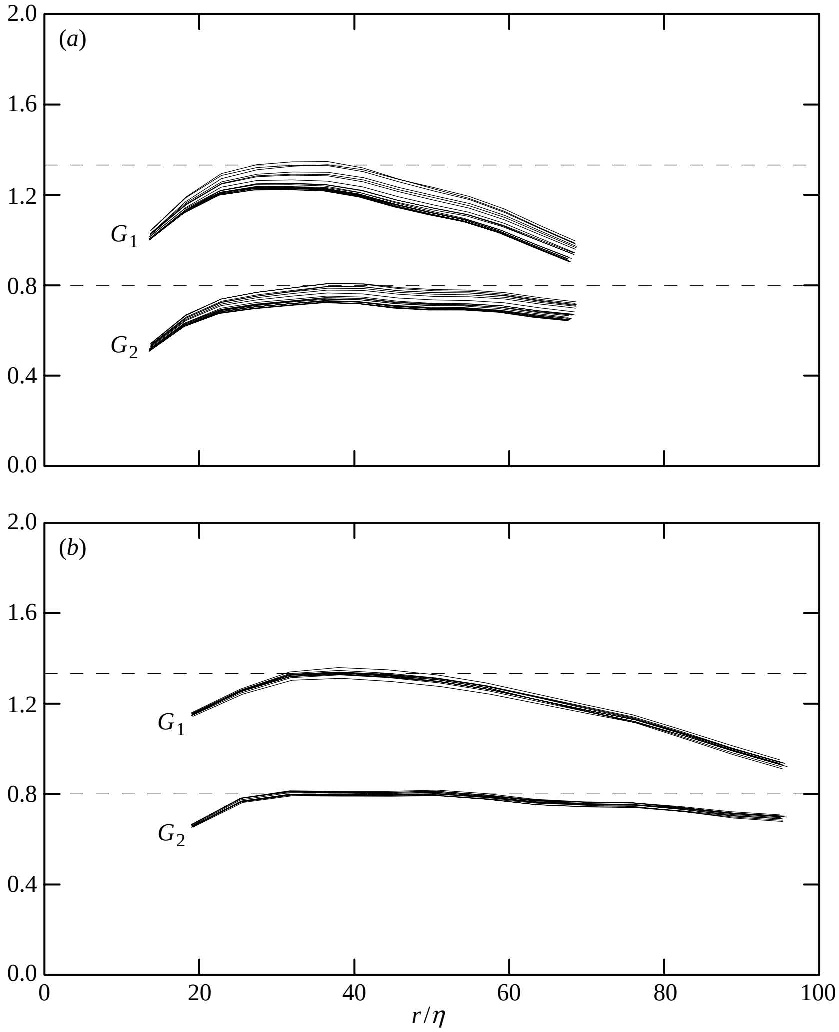

Figure 5. G_1 and G_2 as functions of r/η at different T for $y=(r/\sqrt{3})(1,1,1)$ for forcing (b). $\frac{4}{3}$ or $\frac{4}{5}$ (– – –). (*a*) $\nu=0.001$ and $T=10+i\times 0.5$ for $i=0,\ldots,18$; (*b*) $\nu=0.000\,667$ and $T=9+i\times 0.5$ for $i=0,\ldots,10$. Note convergence for large enough T.

solid angle averages that cannot be computed with currently available computer power, the computational results up to $Re_\lambda=155$ suggest that, in some directions, one can observe an approximate linear scaling regime for $\tilde{S}_3^c$ and $\tilde{S}_{3,K}^c$ that is consistent with the rigorous large Reynolds-number-limiting results for $\tilde{S}_3$ and $S_{3,K}$. However, there also exist other directions for which such agreement does not exist, at least up to $Re_\lambda=155$.

5. Discussion

We conclude by noting that the rigorous equality for $\tilde{S}_3$ and $S_{3,K}$ holds for non-isotropic or inhomogeneous flows as well, since the definition used here involves a space-time solid angle averaging. Our computations, even for a highly symmetric flow

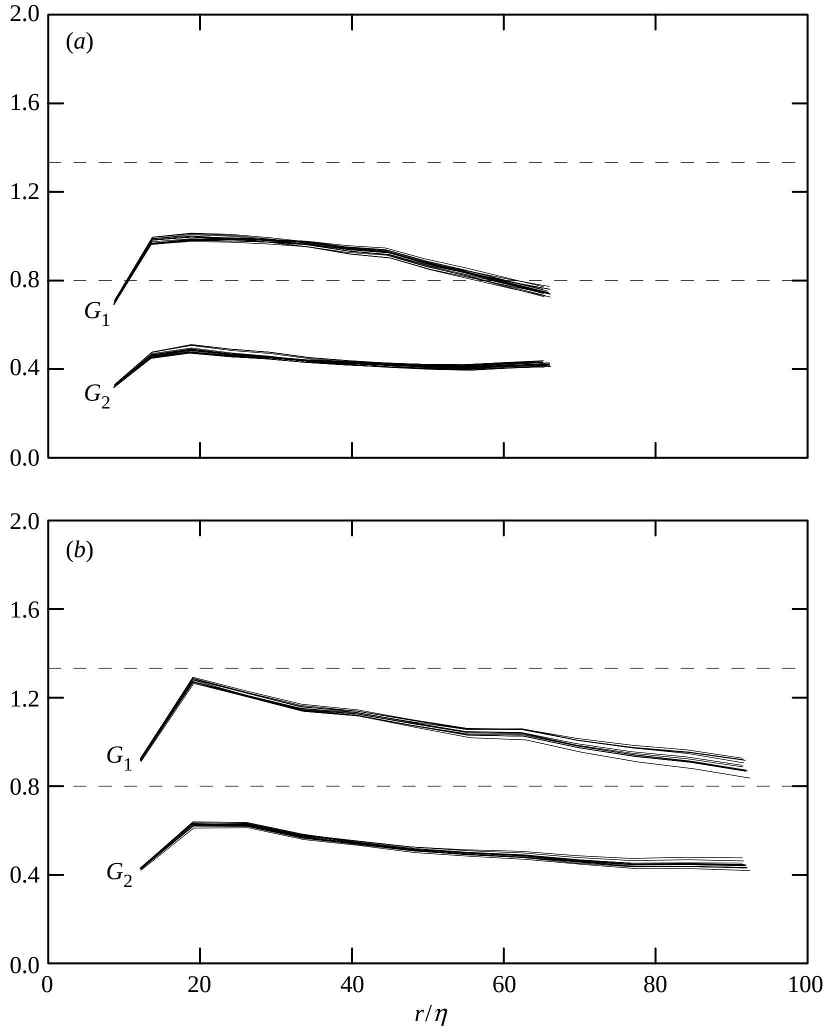

Figure 6. G_1 and G_2 as functions of r/η at different T for $y = (r, 0, 0)$ for forcing (b). $\frac{4}{3}$ or $\frac{4}{5}$ (– – –). (*a*) $\nu = 0.001$ and $T = 10 + i \times 0.5$ for $i = 0, \ldots, 18$; (*b*) $\nu = 0.000\,667$ and $T = 9 + i \times 0.5$ for $i = 0, \ldots, 10$. Note convergence for large enough T.

in a periodic box for a Taylor Reynolds number of up to 155, suggest that the assumptions on isotropy are generally not satisfied. Because of prohibitive computational expense, we are unable to assess, numerically, how the predicted scaling laws in the theoretical large Reynolds number limit hold for the newly defined third-order structure functions (involving space-time solid angle averaging) at the highest Reynolds number for which computation is feasible. However, by dropping solid angle averaging for computational purposes, we noted approximate scaling regimes for some orientation of the displacement vector y, though not for others.

At this point, two possible explanations exist for our computational results. Perhaps, even for larger Taylor Reynolds numbers (beyond what could currently be computed), all anisotropy would disappear. A second explanation would be that anisotropy persists but that computed solid angle averages, as in theory, would have resulted in much closer agreement with the theoretical limiting results (the four-thirds law or the Kolmogorov four-fifths law) even at $Re_\lambda = 155$. Unfortunately, we are unable to distinguish between these two scenarios.

While the numerical computation is, as yet, not practical for flows that are not isotropic, the relations in this paper may prove useful to experimentalists as well as to theoreticians seeking to model the Navier–Stokes dynamics with simpler equations.

S.T. is particularly indebted to Peter Constantin for many helpful discussions and for sharing a preprint on a Littlewood–Paley spectrum in two-dimensional turbulence. S.T. also thanks Leo Kadanoff for discussions and for providing me with the opportunity to be at the University of Chicago. S.T. also benefited very much from helpful discussions with Philip Saffman, Dale Pullin, Paul Dimotakis and Charlie Doering. S.T. was supported by the National Science Foundation (DMS-9500986), the Department of Energy (DE-FG02-92ER25119) and the University of Chicago MRSEC under award number DMR-9400379. Some of the computations were performed at Para//ab of University of Bergen, Norway. We also thank Professor Petter Bjorstad for his help on the computing source.

References

Benzi, R., Ciliberto, S., Tripiccione, R., Baudet, C., Massaioli, F. & Succi, S. 1993 Extended self-similarity in turbulent flows. *Phys. Rev.* E **48**, R29.

Bhattacharjee, A., Ng, C. S. & Wang, X. 1995 *Phys. Rev.* E **52**, 5110.

Boratav, O. N. & Pelz, R. B. 1994 Direct numerical simulation of transition to turbulence from a high-symmetry initial condition. *Phys. Fluids* **6**, 2757.

Boratav, O. N., Pelz, R. B. & Zabusky, N. J. 1992 Reconnection in orthogonally interacting vortex tubes: direct numerical simulation and quantifications. *Phys. Fluids* A **4**, 581.

Constantin, P. 1997 The Littlewood–Paley spectrum for 2-D turbulence. *Theor. Comp. Fluid Dynamics* **9**, 183–189.

Constantin, P. & Fefferman, Ch. 1994 Scaling exponents in fluid turbulence, some analytic results. *Nonlinearity* **7**, 41–57.

Constantin, P., Foias, C. & Temam, R. 1985 *Memoirs of the American Mathematical Society* no. 314, V 53.

Constantin, P., Nie, Q. & Tanveer, S. 1999 Bounds on second-order structure functions and energy in turbulence. *Phys. Fluids* A. (In the press.)

Doering, C. R. & Gibbon, J. D. 1995 *Applied analysis of the Navier–Stokes equation.* Cambridge University Press.

Frisch, U. 1995 *Turbulence.* Cambridge University Press.

Gagne, Y. 1993 Etude experimentalé de l'itermittence et des singularités dans le plan complexe en turbulence developpé. Thèse de doctorat Sciences Physiques, Université de Grenoble.

Kida, S. 1985 Three-dimensional periodic flows with high symmetry. *J. Phys. Soc. Japan* **54**, 2132.

Kida, S. & Ohkitani, K. 1992 Spatio-temporal intermittency and instability of a forced turbulence. *Phys. Fluids* A **4**, 1018.

Kida, S., Yamada, M. & Ohkitani, K. 1989 *Route to chaos in a Navier–Stokes flow.* Lecture Notes in Numerical Applied Analysis, vol. 10, p. 31. Japan: Kinokuniya

Kolmogorov, A. N. 1941*a* *Dokl. Akad. Nauk* **30**, 229.

Kolmogorov, A. N. 1941*b* Dissipation of energy in locally isotropic turbulence. *Dokl. Akad. Nauk* **32**, 16–18. (Reprinted in 1991 *Proc. R. Soc. Lond.* A **434**, 15.)

Lieb, E. 1984 *Comm. Math. Phys.* **92**, 473.

Lundgren, T. S. 1982 Strained spiral vortex model for turbulent fine structure. *Phys. Fluids* **25**, 2193.

L'vov, V. & Procaccia, I. 1996 *Phys. Rev.* E **53**, 3468.

Monin, A. S. & Yaglom, A. M. 1975 *Statistical fluid mechanics*, vol. 2. Cambridge, MA: MIT Press.

Nelkin, M. 1994 *Adv. Phys.* **43**, 143.

Pullin, D. I. & Saffman, P. G. 1993 On the Lundgren–Townsend model of turbulent fine scales. *Phys. Fluids* A **5**, 126–145.

Pullin, D. I. & Saffman, P. G. 1996 *Phys. Fluids* A **8**, 3072.

Ruelle, D. 1982 *Comm. Math. Phys.* **87**, 287.

She, Z. S. & Leveque, E. 1994 Universal scaling laws in fully developed turbulence. *Phys. Rev. Lett.* **72**, 336.

Sreenivasan, K. R. & Kailasnath, P. 1993 *Phys. Fluids* A **5**, 512.

von Karman, T. & Howarth, L. 1938 On the statistical theory of isotropic turbulence. *Proc. R. Soc. Lond.* A **164**, 192.

Yeung, P. K., Brasseur J. G. & Wang, Q. 1995 *J. Fluid Mech.* **283**, 43.