Bounds for second order structure functions and energy spectrum in turbulence

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In this paper we derive upper bounds for the second order structure function as well as for the Littlewood–Paley energy spectrum — an average of the usual energy spectrum $E(k)$. While the upper bound results are consistent with a Kolmogorov type dependence on wave number k, the bounds do not involve the usual dissipation rate ϵ. Instead the bounds involve a dissipative quantity $\hat{\epsilon}$ similar to ϵ but based on the L^3 average of ∇u. Numerical computations for a highly symmetric flows with Taylor microscale Reynolds numbers up to $R_\lambda = 155$ are found to be consistent with the proposition that a relation in the inertial regime of the type $E(k) \sim \hat{C}\hat{\epsilon}^{2/3}k^{-5/3}$ holds with constant $\hat{C}$.
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I. INTRODUCTION

Structure functions and the energy spectrum of turbulent fluids have been studied for more than 5 decades since Kolmogorov's seminal paper.[1] A comprehensive account of the earlier developments is given in Ref. 2. The classical theory states that there exists a range of physical length scales that extends from a small viscous dissipation scale η to a large integral scale L. The energy cascades down in this inertial range from the large scale L to η; the rate ϵ of dissipation of energy is constant, and this constant is the only parameter in this range. For length scales smaller than η viscosity effects are dominant. These two ideas and simple dimensional analysis have been used by Kolmogorov to produce remarkable predictions. First, because it is determined by ϵ and the kinematic viscosity ν alone, the viscous dissipation scale is given by $\eta = \nu^{3/4}\epsilon^{-1/4}$. Second, because they can depend on ϵ alone, the structure functions $S_m(r) = \langle |u(x+r,t) - u(x,t)|^m \rangle$ scale like

$$S_m(r) \sim (\epsilon r)^{m/3}$$

in the inertial range. Here $\langle \cdots \rangle$ is the ensemble average and $u(x,t)$ represents the velocity of the fluid recorded at the point x in space and time t. In particular

$$S_2(r) \sim (\epsilon r)^{2/3}.$$

If the statistics are homogeneous (translation invariant) then the energy spectrum obeys

$$E(k) \sim \epsilon^{2/3} k^{-(5/3)},$$

in the inertial range of wave numbers $k \in [1/L, 1/\eta]$. The Navier–Stokes equations that govern incompressible fluids have not been invoked for these predictions. The Kolmogorov–Obukhov spectrum has been reproduced synthetically from certain vortical structures.[3,4] The first connection to the Navier–Stokes equations appears when one computes the average dissipation of energy in a homogeneous ensemble of solutions:

$$\epsilon = \nu \langle |\nabla u(x,t)|^2 \rangle.$$

The Kolmogorov theory captures experimentally verified truth, and perhaps this is more remarkable than the fact that small corrections to the theory might be required. Landau was the first to point out that the universal predictions based on the statistics of $\nu|\nabla u|^2$ do not take into account *intermittency* — the highly irregular distribution of gradients. It has been the belief of a large part of the turbulence community in recent years that this intermittency has as its counterpart *anomalous scaling* — the fact that the structure functions obey scaling laws with exponents $\zeta_m \neq m/3$ for $m \neq 3$. Whether the anomalous scaling exponents are universal,[5] whether universality is hidden in the relationship between different structure functions but is independent of dissipation,[6] or whether the dissipation contains essentially different exponents[7] are currently debated subjects. A comprehensive account of recent developments is given in Ref. 8.

The mathematical theory of the Navier–Stokes equations has produced few points of contact with the physical theories regarding structure functions and spectrum. Invariant measures have been proved to exist[9,10] but without enough control to establish the ''exact'' $\zeta_3 = 1$ relationship. Connections between the dynamical systems interpretation of number of degrees of freedom of turbulent flows and dissipation based viscous length scales have been obtained.[11] Under the assumption of regularity it has been proven[12] that ϵ is bounded above and consequently that $\zeta_2 \geq 2/3$ must hold if scaling exists. Inequalities that limit the possible extent of anomalous scaling and link the different spatial and temporal intermittency to anomalous scaling exponents have been established.[13] The Littlewood–Paley spectrum was introduced in the context of two dimensional Navier–Stokes turbulence[14] and it was proved to be bounded above by the Kraichnan spectrum. The derivation in that case was without any assumptions. In the present paper we show that similar steps result in upper bounds consistent with a Kolmogorov–Obukhov type Littlewood–Paley energy spectrum. We also prove bounds on second order structure functions that are similar to the Kolmogorov–Obukhov type scaling. In both cases, however, our bounds do not involve the usual dissipation ϵ but a quantity $\hat{\epsilon}$, based on the L^3 average of ∇u which we assume to be finite. Numerical results lend support to the proposition that the ratio $\hat{\epsilon}/\epsilon$ is close to a nonzero constant independent of the forcing.

II. EQUATIONS, ASSUMPTIONS, RESULTS

We start from the incompressible Navier–Stokes equations

$$\begin{aligned}&(\partial_t + u\cdot\nabla - \nu\Delta)u + \nabla p = f,\\ &\nabla\cdot u = 0\end{aligned} \tag{1}$$

in three space dimensions, $x\in D\subset \mathbf{R}^3$. Our numerical calculations are performed with periodic boundary conditions; the theory works as well for the case of decay at infinity. In this paper we will refer only to the periodic case. We assume the period to be L and D to be cube of side L. We will assume without loss of generality that the spatial average of the velocity vanishes. The finite difference

$$(\delta_y u)(x,t) = u(x-y,t) - u(x,t) \tag{2}$$

is the main fluctuating variable; we are interested in long time averages of functionals of $\delta_y u$. We will use the Littlewood–Paley decomposition in this context. The function u can be represented by a Fourier series

$$u(x,t) = \sum_{j\neq 0} u_j(t) e^{i(2\pi/L)(j\cdot x)},$$

where $j\in \mathbf{Z}^3$ is a vector of integer indices, $u_j\in \mathbf{C}^3$ is a time dependent vector of complex components, $j\cdot u_j = 0$ represents incompressibility and $(u_j)^* = u_{-j}$ reality. The Littlewood–Paley decomposition is

$$u(x,t) = u_{(-\infty)}(x,t) + \sum_{m=0}^{\infty} u_{(m)}(x,t),$$

where

$$u_{(-\infty)}(x,t) = L^{-3}\int_{\mathbf{R}^3} \Phi\left(\frac{y}{L}\right) u(x-y,t)\,dy, \tag{3}$$

$$u_{(m)}(x,t) = L^{-3}\int_{\mathbf{R}^3} \Psi_{(m)}\left(\frac{y}{L}\right) u(x-y,t)\,dy, \tag{4}$$

and the functions Φ and $\Psi_{(m)}$ are defined via their Fourier transforms

$$\phi(\xi) = \int_{\mathbf{R}^3} e^{-i(\xi\cdot z)}\Phi(z)\,dz,$$

$$\psi_{(m)}(\xi) = \int_{\mathbf{R}^3} e^{-i(\xi\cdot z)}\Psi_{(m)}(z)\,dz.$$

The function $\phi(\xi)$ is taken to be non-negative, nonincreasing, smooth, radially symmetric, identically equal to 1 for $|\xi|\leqslant \frac{5}{8}$ and identically equal to zero for $|\xi|\geqslant \frac{3}{4}$. The function $\psi_{(0)}(\xi)$ is defined by $\psi_{(0)}(\xi) = \phi(\xi/2) - \phi(\xi)$ and the functions $\psi_{(m)}(\xi) = \psi_{(0)}(2^{-m}\xi)$. Note that from the definitions it follows that $\psi_{(m)}$ is non-negative, equals identically one in an interval $\xi\in[\frac{3}{4}2^m, \frac{5}{4}2^m]$ and vanishes identically outside the interval $\xi\in[\frac{5}{8}2^m, \frac{3}{2}2^m]$. Clearly, also from the definitions it follows that

$$u_{(-\infty)}(x,t) = \sum_{j\neq 0} \phi(j) u_j(t) e^{i(2\pi/L)(j\cdot x)}, \tag{5}$$

and

$$u_{(m)}(x,t) = \sum_{j\neq 0} \psi_{(m)}(j) u_j(t) e^{i(2\pi/L)(j\cdot x)}. \tag{6}$$

Because the spatial average of $\Psi_{(m)}$ vanishes it follows that

$$u_{(m)}(x,t) = \int_{\mathbf{R}^3} \Psi_{(m)}\left(\frac{y}{L}\right) \delta_y(u)(x,t)\,\frac{dy}{L^3} \tag{7}$$

holds for all m.

From the Navier–Stokes equation we deduce the equation obeyed by $\delta_y u$:

$$(\partial_t + u\cdot\nabla - \nu\Delta)\delta_y u + \nabla\delta_y p = \delta_y f + \partial_{y_j}(\delta_y u_j \delta_y u), \tag{8}$$

where

$$\begin{aligned}&\delta_y p = p(x-y,t) - p(x,t),\\ &\delta_y f = f(x-y,t) - f(x,t).\end{aligned} \tag{9}$$

For any quantity $Q = Q(x,t)$ we will use $\langle Q\rangle$ to denote the space–time average

$$\langle Q\rangle = \limsup_{T\to\infty} \frac{1}{T}\int_0^T L^{-3}\int_D Q(x,t)\,dx\,dt.$$

We will consider given body forces that are regular enough

$$\langle |f|^2\rangle + \langle |\nabla f|^2\rangle < \infty. \tag{10}$$

The assumption of this paper is

$$\langle |\nabla u|^3\rangle < \infty. \tag{11}$$

This is a very strong assumption and is not known to be true for arbitrary solutions of the Navier–Stokes equations. The assumption implies regularity. The physical parameters associated with the quantity above are

$$\hat{\epsilon} = \nu\{\langle |\nabla u|^3\rangle\}^{2/3}, \tag{12}$$

the corresponding dissipative scale

$$\hat{\eta} = \nu^{3/4}(\hat{\epsilon})^{-1/4}, \tag{13}$$

and the corresponding dissipative cutoff wave number

$$\hat{k}_d = \nu^{-3/4}(\hat{\epsilon})^{1/4}.$$

The traditional ϵ is

$$\epsilon=\nu\langle|\nabla u|^2\rangle. \tag{14}$$

Our main results are

$$E_{\mathrm{LP}}(k)\leqslant C_\psi(\hat{\epsilon})^{2/3}k^{-(5/3)}\left(\frac{k}{\hat{k}_d}\right)^{-10/3} \tag{15}$$

[see (17) for the definition of the spectrum] and

$$S_2(r)\leqslant(\epsilon r)^{2/3}\left(\frac{r}{\eta}\right)^{4/3}\left[\frac{1}{3}+\frac{r}{12\lambda}\right]+\frac{1}{8}(\hat{\epsilon}r)^{2/3}\left(\frac{r}{\hat{\eta}}\right)^{10/3} \tag{16}$$

[see (27) for the definition of $S_2(r)$ and (38) the definition of λ].

III. BOUNDS ON THE SPECTRUM

Consider the mth Littlewood–Paley component of the velocity, We define the Littlewood–Paley spectrum[14] to be

$$E_{\mathrm{LP}}(k)=k_m^{-1}\langle|u_{(m)}|^2\rangle \tag{17}$$

for $k_{m-1}\leqslant k<k_m$, $m\geqslant 1$ with $k_m=2^mL^{-1}$. Note that, in view of the Plancherel theorem this is just

$$E_{\mathrm{LP}}(k)=k_m^{-1}\sum_{j\neq 0}|\psi_{(m)}(j)|^2\langle|u_j(t)|^2\rangle. \tag{18}$$

From the Navier–Stokes equation we obtain the evolution equation of the Littlewood–Paley components (see Ref. 14 for the two dimensional analogue)

$$(\partial_t+u\cdot\nabla-\nu\Delta)u_{(m)}+\nabla p_{(m)}=W_{(m)}+f_{(m)}, \tag{19}$$

where $p_{(m)}$, $f_{(m)}$ are the Littlewood–Paley components of the pressure and force and

$$W_{(m)}(x,t)=\int_{\mathbf{R}^3}\Psi_{(m)}\left(\frac{y}{L}\right)\partial_{y_j}[\delta_y(u_j)(x,t)\delta_y(u)(x,t)]\frac{dy}{L^3}. \tag{20}$$

We multiply (19) by $u_{(m)}$, integrate dx and take a long time average. We obtain

$$\nu\langle|\nabla u_{(m)}|^2\rangle=\langle(W_{(m)}+f_{(m)})u_{(m)}\rangle. \tag{21}$$

We will take m so that $2^{m-1}\geqslant C$ for some $C\geqslant 1$ and assume for simplicity that $f_{(m)}=0$. This amounts to the condition that the force has Fourier transform supported in a ball, corresponding to the physical condition of large scale forcing. The effects of small scale forcing can be easily investigated also. The left hand side of (21) obeys quite obviously

$$\nu\langle|\nabla u_{(m)}|^2\rangle\geqslant\nu k_m^2\langle|u_{(m)}|^2\rangle=\nu k_m^3E_{\mathrm{LP}}(k_m). \tag{22}$$

We estimate now the right hand side of (21). First we note that

$$\int_D W_{(m)}(x,t)u_{(m)}(x,t)dx$$

$$=\int\Gamma_{(m),j}\frac{dy\,dz}{L^3L^3}\delta_y(u_j)(x,t)\delta_y(u)(x,t)\delta_z(u)(x,t)dx$$

with

$$\Gamma_{(m),j}(y,z)=-\partial_{y_j}\left[\Psi_{(m)}\left(\frac{y}{L}\right)\right]\Psi_{(m)}\left(\frac{z}{L}\right)$$

follows directly from the definitions, the expression (20) and one integration by parts in the y integral. Using straightforward calculus inequalities we note that

$$\int_D|\delta_y(u)(x,t)|^2|\delta_z(u)(x,t)|dx\leqslant|y|^2|z|\int_D|\nabla u(x,t)|^3\,dx$$

and therefore it follows that

$$\left|\int_D W_{(m)}(x,t)u_{(m)}(x,t)dx\right|\leqslant C_\psi k_m^{-2}\int_D|\nabla u(x,t)|^3\,dx \tag{23}$$

where

$$C_\psi=\int\int|\nabla\Psi_{(0)}(a)||a|^2|\Psi_{(0)}(b)||b|da\,db. \tag{24}$$

We deduce from (21), (22) and (23) that

$$E_{\mathrm{LP}}(k)\leqslant C_\psi k^{-5}\nu^{-1}\langle|\nabla u|^3\rangle. \tag{25}$$

Using the definitions of $\hat{\epsilon}$ and $\hat{k}_d$ and the bound (25) we have proven thus:

Theorem 1: Consider three-dimensional body forces that satisfy

$$\hat{f}(k)=0$$

for all $|k|\geqslant C/L$ and some $C>0$. Consider solutions of the three dimensional Navier–Stokes equation that satisfy $\hat{\epsilon}<\infty$. Then

$$E_{\mathrm{LP}}(k)\leqslant C_\psi(\hat{\epsilon})^{2/3}k^{-(5/3)}\left(\frac{k}{\hat{k}_d}\right)^{-(10/3)}$$

holds for $|k|\geqslant C/L$.

IV. STRUCTURE FUNCTION

Consider the spatial average

$$s_2(y,t)=\frac{1}{L^3}\int_D|\delta_yu(x,t)|^2\,dx. \tag{26}$$

The long time average of this quantity is the traditional second structure function. We perform a solid angle average and define, as in Ref. 15

$$S_2(r)=\limsup_{T\to\infty}\frac{1}{T}\int_0^T\frac{1}{4\pi}\int_{|\hat{y}|=1}s_2(r\hat{y},t)dS(\hat{y})dt. \tag{27}$$

From (8) it follows after multiplication and integration that

$$(\partial_t-2\nu\Delta)s_2+4\epsilon(t)=\partial_{y_j}s_{3,j}(y,t)+2F(y,t), \tag{28}$$

where

$$F(y,t)=\frac{1}{L^3}\int_D\delta_yf(x,t)\cdot\delta_yu(x,t)dx, \tag{29}$$

$$s_{3,j}(y,t)=\frac{1}{L^3}\int_D\delta_yu_j(x,t)|\delta_yu(x,t)|^2dx, \tag{30}$$

$$\epsilon(t) = \nu \frac{1}{L^3} \int_D |\nabla u(x,t)|^2 dx. \tag{31}$$

This is the analogue of the familiar[2] anisotropic generalization of the von Karman–Howarth equation. The spherical average of $s_2(y,t)$ is $S_{(2)}(r,t)$:

$$S_{(2)}(r,t) = \frac{1}{4\pi} \int_{|\hat{y}|=1} s_2(r\hat{y},t) dS(\hat{y}). \tag{32}$$

A volume integral of (28) on $|y| \leqslant r$ results in

$$\frac{\partial}{\partial r} S_{(2)}(r,t) = 2\frac{r\epsilon(t)}{3\nu} - \frac{1}{2\nu}\tilde{S}_3(r,t) - \frac{1}{\nu 4\pi r^2} \times \int_{|y|\leqslant r} \left(F(y,t) - \frac{1}{2}\partial_t s_2(y,t) \right) dy, \tag{33}$$

where

$$\tilde{S}_3(r,t) = \frac{1}{4\pi} \int_{|\hat{y}|=1} \hat{y}_j s_{3,j}(r\hat{y},t) dS(\hat{y}). \tag{34}$$

The inequality

$$|\tilde{S}_3(r,t)| \leqslant \frac{r^3}{L^3} \int_D |\nabla u(x,t)|^3 \, dx$$

is straightforward from definitions. Similarly simple is the inequality

$$|F(y,t)| \leqslant |y| \|\nabla f(\cdot,t)\| \sqrt{s_2(y,t)},$$

where $\|\nabla f(\cdot,t)\| = \sqrt{L^{-3}\int_D |\nabla f(x,t)|^2 \, dx}$ is the L^2 norm. Integrating dr and time averaging we obtain

$$S_2(r) \leqslant \frac{r^2}{3\nu}\epsilon + \frac{r^4}{8}\nu^{-(5/2)}(\hat{\epsilon})^{3/2} + \frac{r^3}{12\nu} UG, \tag{35}$$

where

$$G = \sqrt{\limsup_{T\to\infty} \frac{1}{T} \int_0^T L^{-3} \int_D |\nabla f(x,t)|^2 \, dx \, dt} \tag{36}$$

and

$$U = \sup_{0\leqslant r \leqslant L} \sqrt{S_2(r)}. \tag{37}$$

The quantity U can be proved to be finite as a consequence of the assumption $\hat{\epsilon} < \infty$. The inequality (35) is the main result of this section. Let us define the length λ by

$$\lambda = \frac{\epsilon}{UG}. \tag{38}$$

We have proved therefore:

Theorem 2: Consider solutions of the Navier–Stokes equations that satisfy the assumption $\hat{\epsilon} < \infty$. Then

$$S_2(r) \leqslant (\epsilon r)^{2/3} \left(\frac{r}{\eta}\right)^{4/3} \left[\frac{1}{3} + \frac{r}{12\lambda}\right] + \frac{1}{8}(\hat{\epsilon} r)^{2/3} \left(\frac{r}{\hat{\eta}}\right)^{10/3}$$

holds for all r.

V. NUMERICAL CALCULATIONS

The rigorous results derived above are all in the form of inequalities. We resort to direct numerical simulations to determine whether these rigorous inequalities suggestive of a Kolmogorov type energy spectrum are consistent with a large Reynolds number direct numerical simulation. We would like to determine the ratio $\hat{\epsilon}/\epsilon$ for different forcings and Reynolds numbers and whether the power law (15) describes accurately the transition to dissipative scales in the spectrum. Unfortunately, some of these questions cannot be answered in a definitive manner now because of limitations of computational power. The results given in this section, therefore, have to be taken as indicative only of trends.

As the Littlewood–Paley energy spectrum averages on progressively larger intervals in k for large m, while computer calculations are not feasible for a very large range of k, we confined ourself to the calculation of the averaged energy spectrum, defined by

$$\bar{E}(k) = \frac{1}{2} \lim_{T\to\infty} \int_0^T \frac{dt}{T} \sum_{k-\frac{1}{2} \leqslant |\mathbf{k}| < k+\frac{1}{2}} |\hat{u}(\mathbf{k})|^2, \tag{39}$$

where $\hat{u}(\mathbf{k})$ is the coefficient of a Fourier series of u.

As before[15] in a different context, we solve the Navier–Stokes equation by forcing $f(x,t)$ in a 2π-periodic cube $(L=2\pi)$ with an initial condition of a "*high symmetry*" as discussed in Ref. 16. This was also used in a prior calculation.[15] In particular, the flow at all times admits the following Fourier expansion for x_1 component of the velocity:

$$u_1(x_1,x_2,x_3,t) = \left(\sum_{\text{even } l,m,n=0}^{\infty} + \sum_{\text{odd } l,m,n=1}^{\infty} \right) \hat{u}_{1\{l,m,n\}} \times (t) \sin lx_1 \cos mx_2 \cos nx_3. \tag{40}$$

The other velocity components are determined by a permutation symmetry $u_1(x_1,x_2,x_3) = u_2(x_2,x_3,x_1) = u_3(x_3,x_1,x_2)$. The special structure of the Fourier components in (40) and the permutation relationship above saves computational time and memory.[16,17] In our study, the initial condition is chosen to be the same as that in Ref. 18. Specifically,

$$u_1(x_1,x_2,x_3,t=0) = \sin x_1 (\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3). \tag{41}$$

The numerical method for solving the incompressible Navier–Stokes equation is based on a Fourier pseudospectral technique. The details can be found in Refs. 17 and 21. To numerically calculate ϵ, $\hat{\epsilon}$, we use a second order Adams–Bashforth method. The spatial integrations in x are evaluated through summation over N evenly spaced grid points in the 2π-periodic box, where N is the number of grid points in each direction of the 2π periodic domain. This quadrature is spectrally accurate.

We present results for three different forcing functions and two different values of viscosity ν:

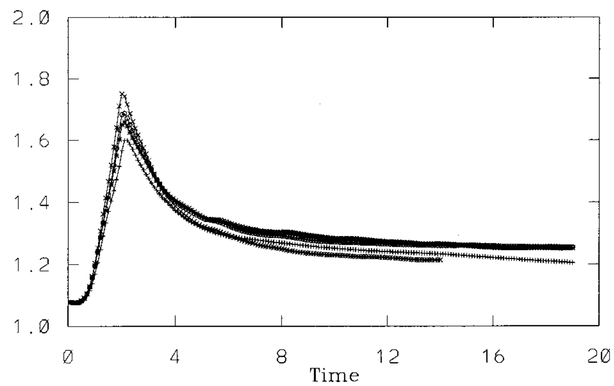

FIG. 1. $\hat{\epsilon}/\epsilon$ vs time: (x) forcing (1) with $\nu=0.000\,667$, $R_\lambda=155$; (o) forcing (1) with $\nu=0.001$, $R_\lambda=134$; (*): forcing (2) with $\nu=0.001$, $R_\lambda=132$; (+) forcing (3) with $\nu=0.000\,667$, $R_\lambda=83$.

(1) $f(x,t)$ is chosen such that the Fourier mode $\hat{u}_{1\{1,3,1\}} = -\hat{u}_{1\{1,1,3\}}=1$ all the time in order to imitate a constant energy supply at lower wave numbers;[18]

(2) $f_1(x,t)=0.2\times u_1(x_1,x_2,x_3,t=0)$;

(3) $f_1(x,t)=0.2\times \sin x_1(\cos 7x_2 \cos x_3 - \cos x_3 \cos 7x_2)$.

With forcing (1), we use (a) $\nu=0.000\,667$ and (b) $\nu=0.001$. However, we only use $\nu=0.001$ for forcing (2) and $\nu=0.000\,667$ by forcing (3). It should be noted that in Ref. 15, we had reported the Navier–Stokes computation for cases (1) and (2). As before, we choose $N=256$ for all cases with time step $\triangle t=0.001$ for $\nu=0.001$ and $\triangle t=0.0005$ for $\nu=0.000\,667$.[18,17] For $f(x,t)=0$, we have tested our computational results against those presented in Ref. 17. For forcing (1), we have compared our results with those studied in Refs. 18 and 20. All computations are performed by using 64 bit arithmetic. Further, the long time average is replaced by a finite T in the expression $(1/T)\int_0^T dt(\dots)$ for T chosen large enough so that further variation with T is small.

We define a Taylor microscale Reynolds number R_λ as

$$R_\lambda=\sqrt{\frac{20}{3}}\frac{\overline{E}}{\sqrt{\nu\epsilon}}, \tag{42}$$

where $\overline{E}$ is the time-averaged energy for a large enough value of T (so that it has equilibrated). In Fig. 1, we plot the ratio $\hat{\epsilon}/\epsilon$ as a function of averaging time T for four different cases where R_λ ranges from 83 to 155. In all the four cases, the ratios start to settle down around $T=10$. Here, the unit of time is implicit by the choice of $L=2\pi$ and taking f, which has units of acceleration, to be given by one of the expressions (1)–(3) above. This value of T for which $\hat{\epsilon}/\epsilon$ settles down is found to be about the same as that for the averaged dissipation rate ϵ and energy $\overline{E}$ (as given in Ref. 15).

In Fig. 2, we plot the following quantity:

$$\frac{\overline{E(k)}}{\hat{\epsilon}^{2/3}k^{-5/3}} \tag{43}$$

in log as functions of $\log(K_1)$ where normalized wave number K_1 is defined as:

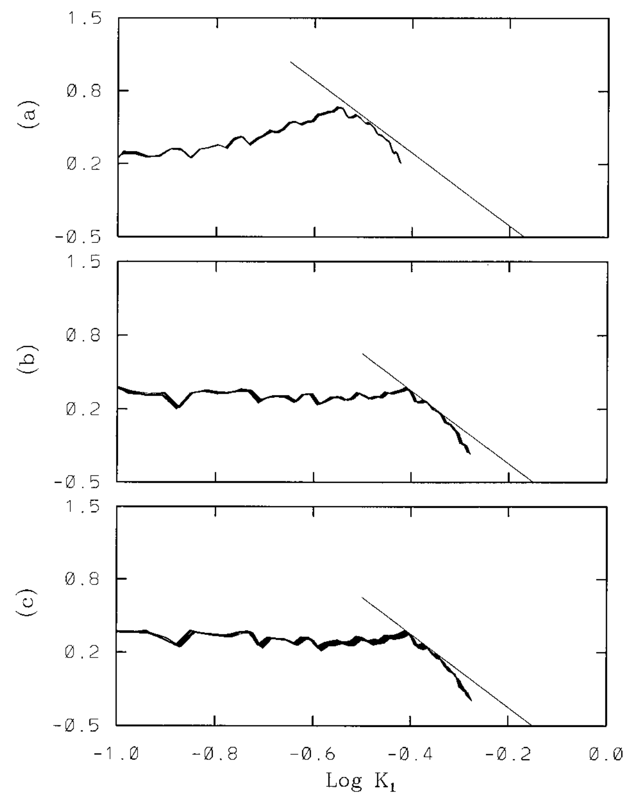

FIG. 2. log (energy spectrum/ $\hat{\epsilon}^{2/3}k^{-5/3}$) as a function of $\log(K_1)$. The straight line has slope $-10/3$. (a) forcing (1) with $\nu=0.000\,667$ at $t=10,\dots,14$ for every $t=0.5$; (b) forcing (1) with $\nu=0.001$ at $t=10,\dots,19$ for every $t=0.5$; (c) forcing (2) with $\nu=0.001$ at $t=10,\dots,19$ for every $t=0.5$.

$$K_1=k\hat{\eta}. \tag{44}$$

We notice that there is a range of K_1 where the curves are relatively flat, suggestive of a Kolmogorov type spectrum in an inertial regime; however, for larger K_1, there is apparently a drop-off characteristic of dissipation scale. In the observed range of K_1, where the curves are flat (suggestive of an inertial regime), we used a least square fit procedure to determine the values of the constants A and A_1 in assumed relations of the form:

$$\overline{E(k)}\sim A\epsilon^{2/3}k^{-5/3} \quad \overline{E(k)}\sim A_1\hat{\epsilon}^{2/3}k^{-5/3}. \tag{45}$$

In Table I, we present the computed values from the fitting. In order to compare cases for different forcing and viscosity, we introduce the normalized wave number based on η, i.e., $K=k\eta$. We noted that a 5% changes of the fitting range of K indicated in Table I does not alter A and A_1 within 0.1.

TABLE I. Least-square fits of the energy spectrum.

	R_λ	A	A_1	Range for K	Range for T
1a	155	1.9	1.7	(0.07,0.16)	(10, 14)
1b	134	2.1	1.8	(0.2,0.37)	(10, 19)
2	132	2.1	1.8	(0.2,0.36)	(10, 19)
3	83	2.3	1.9	(0.2,0.35)	(12, 19)

A rough estimate of the extent of the inertial subrange based upon experimental data suggests that R_λ should be at least 100 to exhibit the power law over 1 or 2 decades of wave number.[20,21] Therefore, the values of A and A_1 for the last case ($R_\lambda = 83$) should perhaps be discounted. By averaging the first three cases and keeping two significant digits, we obtain that $A = 2.0$ and $A_1 = 1.8$. The corresponding values of η/L for the three cases (1a), (1b) and (2) are 0.001 06,0.001 49 and 0.001 52, respectively.

For values of K_1 in Fig. 2, where there is an observed drop-off in the curve (indicative of the transition to the dissipative range), we tried to determine if a power law scaling in the form of the right hand side of (15) is appropriate in the dissipation range. The straight lines in Fig. 2, next to the curve, would be the exact fit corresponding to the power law $-10/3$. It seems that the fitting is consistent in a short range but it is difficult to tell whether a transition to exponential decay is developing.

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