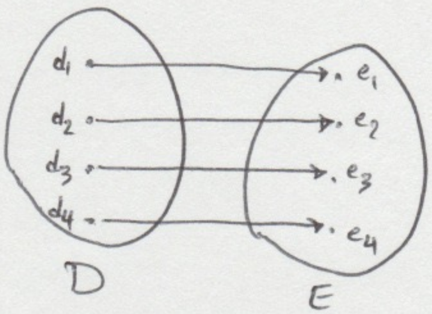
Functions and Models
Inverse Functions and Logarithms

Definition of one-to-one Function:

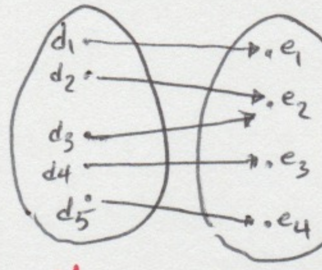
A function f is called a one-to-one function if it never takes on the same value twice; that is:

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2$$

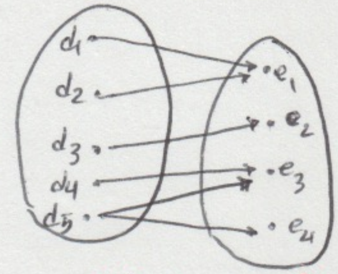
in other words a one-to-one function maps no more than one element in the domain to one element in the range of the function.



one-to-one function



NOT a one-to-one function because d_2 and d_3 are mapped to e_2

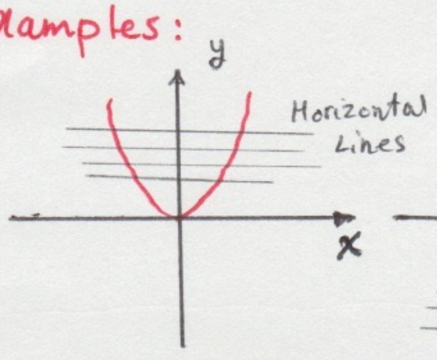


Not a function why?

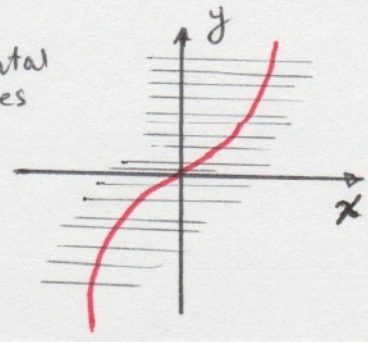
Horizental line Test:

A function is one to one if and only if no horizontal line intersects its graph more than once.

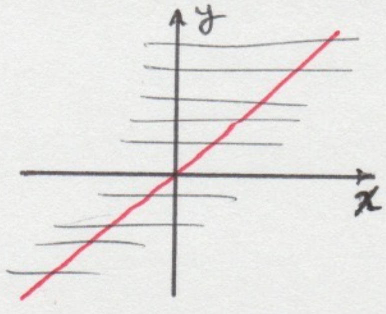
Examples:



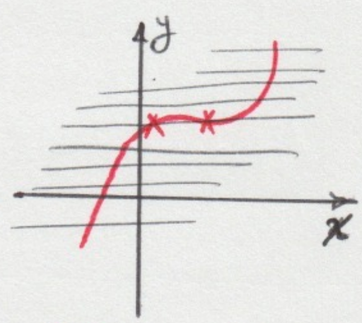
Not one-to-one



one-to-one



One-to-one



Not one-to-one

Example: is the function $f(x) = x^3$ one to one?

if $x_1 \neq x_2$ Then $x_1^3 \neq x_2^3$ (Because Two different # Can not have the Same cube) therefore $f(x) = x^3$ is a one to one function

Example: is function $g(x) = x^2$ one to one?

We plug in the values of $x = 1$ & $x = -1$ in The functiong.

$g(1) = (1)^2 = 1$

$g(-1) = (-1)^2 = 1$

$\Longrightarrow g(x_1) = g(x_2)$ but $x_1 \neq x_2$

Therefore g is Not a one-to-one function.

Inverse function:

Let f be a one-to-one function with domain A and Range B. Then the inverse function f^{-1} has domain B and range A and defined as $f^{-1}(y) = x \Longleftrightarrow f(x) = y$

function f maps $\underline{x}$ into $\underline{y}$

function f^{-1} maps $\underline{y}$ into $\underline{x}$

Remember:

domain of f^{-1} = range of f
range of f^{-1} = domain of f

Cancellation Equations:

when we find an inverse of a function, it is good practice to check to see if we have done it correctly. in order to do so, a simple way is to find $f^{-1}(f(x))$ or $f(f^{-1}(x))$ the results for both of these should be x (the inverse of a function, neutralized what ever the function f has done to it)

$$\Longrightarrow f^{-1}(f(x)) = x$$

$$f(f^{-1}(x)) = x$$

Example: find the inverse of function $f(x) = x^3$ and check to see if your answer is correct?

① $f(x) = x^3 \longrightarrow y = x^3$

② $x = \sqrt[3]{y}$

③ $y = \sqrt[3]{x} \Longrightarrow f^{-1}(x) = \sqrt[3]{x}$

Sanity check $f^{-1}(f(x)) = \sqrt[3]{f(x)} = \sqrt[3]{x^3} = x$ ✓ The answer is correct

Remember: Only One-to-one functions are invertible (have inverse function) WHY? (I explained the answer in The Class)

How to find the inverse of a one-to-one function f:

step 1 write down $y = f(x)$

step 2 Solve this equation for x in terms of y (if possible)

step 3 to express f^{-1} as a function of x, interchange x & y.
The resulting equation is $y = f^{-1}(x)$

Example: find the inverse of function $f(x) = x^3 + 2$?

step 1: $y = x^3 + 2$

step 2: $x^3 = y - 2 \Rightarrow x = \sqrt[3]{y-2}$

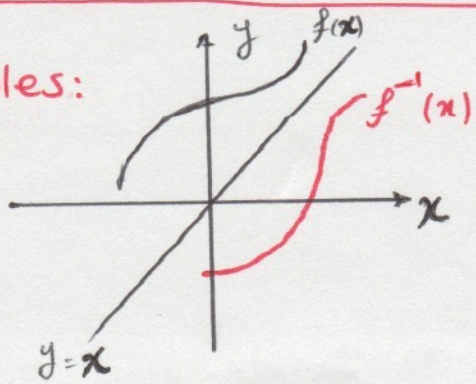
step 3: $y = \sqrt[3]{x-2} \Rightarrow f^{-1}(x) = \sqrt[3]{x-2}$

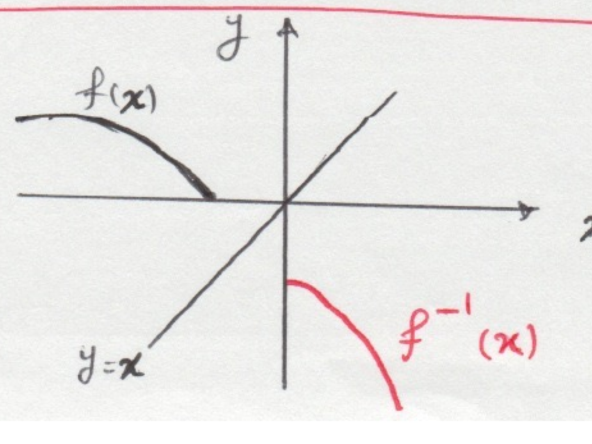
Sanity check: $f^{-1}(f(x)) = x$? $\rightarrow$ $f^{-1}(f(x)) = \sqrt[3]{f(x) - 2}$

$$= \sqrt[3]{(x^3+2) - 2} = \sqrt[3]{x^3} = x \checkmark$$

Important Note: the graph of f^{-1} can be obtained by reflecting the graph of f about the line $y = x$.

Examples:





Logarithmic Functions:

if $a > 0$ and $a \neq 1$ the exponential function $f(x)$ is either increasing or decreasing ($f(x) = a^x$) why?

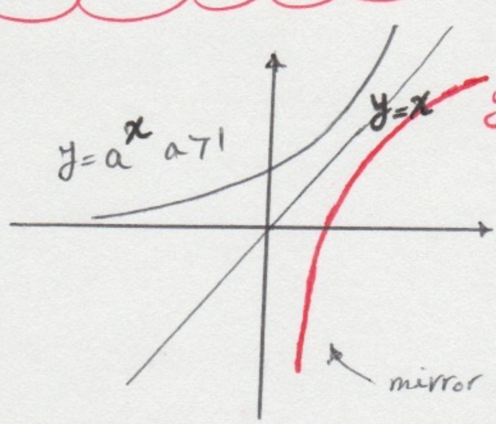
which means it will pass the <u>horizontal Line test</u> therefore, it has an inverse function f^{-1} which is called **Logarithmic function with base a** and is denoted by $\log_a$

if we use the formulation of an inverse function given by previous def. of inverse functions

$$f^{-1}(x) = y \iff f(y) = x$$

$$\Rightarrow \boxed{\log_a x = y \iff a^y = x}$$

Very Very important



mirror reflection of exponential function wrt $y=x$ line.

Laws of Logarithms: If x & y are positive numbers, Then

1) $\log_a(xy) = \log_a x + \log_a y$

2) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

3) $\log_a(x^r) = r\log_a x$ (where r is any real number)

Example:
Prove the first Law of logarithms:

$$\log_a xy = \log_a x + \log_a y$$

Lets assume $\log_a x = z$ & $\log_a y = t$ according to definition:

$$\log_a x = z \Rightarrow x = a^z$$

$$\log_a y = t \Rightarrow y = a^t$$

$$\Rightarrow xy = \underbrace{a^z \cdot a^t = a^{z+t}}_{\text{According to Laws of exponents}}$$

We take the log from both sides $\Rightarrow \log_a xy = \log_a a^{z+t} = z+t$

$$\Rightarrow \boxed{\log_a xy = \log_a x + \log_a y}$$

Example: use the laws of Logarithms to evaluate $\log_2 80 - \log_2 5$?

$$\log_2 80 - \log_2 5 = \log_2 80/5 = \log_2 16 = \log_2 2^4 = 4\log_2 2 = 4$$

Natural Logarithms: Similar to exponential functions that $e \simeq 2.71$ had significant importance, the log function with the base e also has significant importance, and it is denoted by $\underline{\underline{\ln}}$, called **natural** log.

$$\Rightarrow \boxed{\log_e x = \ln x}$$

① $\ln x = y \Rightarrow x = e^y$

② $\begin{cases} \ln e^x = x & x \in \mathbb{R} \\ e^{\ln x} = x & x > 0 \end{cases}$

③ $\ln e = 1$

Example: find x if $\ln x = 5$?

$$\ln x = 5 \Rightarrow x = e^5$$

Example: Solve the equation $e^{5-3x} = 10$?

$$\ln e^{5-3x} = \ln 10 \Rightarrow 5 - 3x = \ln 10 \Rightarrow$$

$$\boxed{x = \frac{5 - \ln 10}{3}}$$

Change of base formula: The following formula can be used to express logarithms with any base in terms of natural log

$$\log_a x = \frac{\ln x}{\ln a}$$ → prove This @ home

Example: $\log_8 5 = \frac{\ln 5}{\ln 8}$